لا يك
هذا

أحسب مفكوك لوران للدالة $f(z)=\dfrac{1}{z^2-4z+3}$ في النطاقات:

(i) $1<|z|<3$ (ii) $|z|<1$ (iii) $|z|>3$

الحل:

$$\frac{1}{1-u}=\sum_{n=0}^{\infty}u^n \; ; \; |u|<1$$

$$f(z)=\frac{1}{(z-1)(z-3)}=\frac{A}{z-1}+\frac{B}{z-3}$$

$$A(z-3)+B(z-1)=1 \Rightarrow z=1 \to A=-\tfrac{1}{2} \;,\; z=3 \to B=\tfrac{1}{2}$$

$$f(z)=\frac{-\frac{1}{2}}{z-1}+\frac{\frac{1}{2}}{z-3}$$

(i)

$$1<|z| \Rightarrow \left|\frac{1}{z}\right|<1 \Rightarrow \frac{1}{z-1}=\frac{1}{z(1-\frac{1}{z})}=\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1 \Rightarrow \frac{1}{z-3}=\frac{1}{-3(1-\frac{z}{3})}=-\frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n=-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$

$$\Rightarrow f(z)=-\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$$

(ii)

$$f(z)=\frac{1/2}{1-z}+\frac{\frac{1}{2}}{-3(1-\frac{z}{3})} \qquad \left(|z|<1 \Rightarrow \left|\frac{z}{3}\right|<1\right)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}z^n-\frac{1}{2.3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n=\sum_{n=0}^{\infty}\frac{1}{2}z^n-\sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{z^n}{3^{n+1}}$$

$$=\sum_{n=0}^{\infty}\left[\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3^{n+1}}\right]z^n$$

(iii)

$$|z|>3>1 \Rightarrow \left|\frac{1}{z}\right|<\frac{1}{3}<1$$

$$\left|\frac{z}{3}\right|>1 \Rightarrow \left|\frac{3}{z}\right|<1$$

$$f(z)=\frac{-\frac{1}{2}}{z(1-\frac{1}{z})}+\frac{\frac{1}{2}}{z(1-\frac{3}{z})}$$

$$=\frac{-1}{2z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{1}{2z}\cdot\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{-\frac{1}{2}}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{3^n}{z^{n+1}}=\sum_{n=0}^{\infty}\left[-\frac{1}{2}+\frac{3^n}{2}\right]\cdot z^{-(n+1)}$$