MATH107 Vectors and Matrices

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Gauss-Jorden elimination Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

Ex.4

$x_1 + x_2 + 2x_3 = 8$

$-x_1 - 2x_2 + 3x_3 = 1$

$3x_1 - 7x_2 + 4x_3 = 10$

Solution:

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow{(\mathbf{R_1+R_2, -3R_1+R_3})} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

$$\xrightarrow{(\mathbf{-R_2, 10R_2+R_3})} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \xrightarrow{(\mathbf{-R_3/52})} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{(\mathbf{-2R_3+R_1, 5R_3+R_2})} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{(\mathbf{-R_2+R_1})} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

Row Echelon Form

- "1" (leading entry) must be in the beginning of each row.
- "1" must be on the right of the above leading entry.
- Below the leading entry all values must be zero.
- A row containing all zero values must be in the bottom.

Examples

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- "1" (leading entry) must be in the beginning of each row.
- "1" must be on the right of the above leading entry.
- All leading entries in the column containing leading enty must be zero.
- A row containing all zero values must be in the bottom.

Examples

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Conditions on Solutions

Example 7
For which values of $'a'$ will be following system

$$x + 2y - 3z = 4 \quad (1)$$
$$3x - y + 5z = 2 \quad (2)$$
$$4x + y + (a^2 - 14)z = a + 2 \quad (3)$$

(i) infinity many solutions?
(ii) No solution?
(iii) Exact one solution?
Solution:
Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow{(\mathbf{R_2-3R_1, R_3-4R_1})} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

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$$\xrightarrow{(-\frac{1}{7}\mathbf{R_2},\mathbf{R_3}-\mathbf{R_2})} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

The equivalent linear system form is:

$$x + 2y - 3z = 4 \quad (4)$$

$$y - 2z = \frac{10}{7} \quad (5)$$

$$(a^2 - 16)z = a - 4. \quad (6)$$

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Case I: when $a = 4$, then $0z = 0$. So the linear system is:

$$\begin{aligned} x + 2y - 3z &= 4 \\ y - 2z &= \frac{10}{7}. \end{aligned}$$

As the number of equations are less than the number of the variables (unknowns), then there are infinite solutions. Choosing $z = t$, where $t \in \mathbb{R}$, then the solution is

$$\begin{aligned} z &= t \\ y &= \frac{10}{7} + 2t \\ x &= 4 + 3t - 4t - \frac{20}{7} = -t + \frac{8}{7}. \end{aligned}$$

Case II: When $a = -4$, then $0z = -8$, therefore, there is no solution.

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Case III: When $a \neq \pm 4$, we have only one solution. For example when $a = 0$, the solution is:

$$\begin{aligned} z &= \frac{1}{4} \\ y &= \tfrac{10}{7} - 2(\frac{1}{4}) = \frac{13}{14} \\ x &= 4 - 2(\frac{13}{14}) + 3(\frac{1}{4}) = \frac{39}{28}. \quad \text{(check!!)} \end{aligned}$$

Case III: When $a \neq \pm 4$, we have only one solution. For example when $a = 0$, the solution is:

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Q: For what value of λ does the system of equations have

(i) have infinitely many solutions?

(ii) have no solution?

(iii) have just one solution?.

$$\begin{aligned} 3x + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= 3. \end{aligned}$$

Ex3

$$x + 2y - 3z = 5 \quad (7)$$

$$3x - y = 3 \quad (8)$$

$$2x + 4z = -1 \quad (9)$$

Solution:

Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 3 & -1 & 0 & 3 \\ 2 & 0 & 4 & -1 \end{bmatrix} \xrightarrow{(\mathbf{R_2 - 3R_1, R_3 - 2R_1})} \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -7 & 9 & -12 \\ 0 & -4 & 10 & -11 \end{bmatrix}$$

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$$z = -0.85$$

$$y = 0.617$$

$$x = 1.205$$

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Homogeneous system

A system of equation of the form

$$AX = 0.$$

1. The homogeneous system has solutions, $x_1 = x_2 = x_3 = \cdots = 0$ (trivial solution)
2. The homogeneous system has infinitely many non-trivial solutions and trivial solutions.
3. The homogeneous system has a non-trivial solution if and only if A is a singular matrix $(|A| = 0)$.