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4.3) Buhlmann-strub model

year 1 — $X_{11}, X_{12}, \dots, X_{1m_1}$ (m_1 Policies)

year 2 — $X_{21}, X_{22}, \dots, X_{2m_2}$

...

year n — $X_{n1}, \dots, X_{n, m_n}$ (m_n Policies)

$$X_1 = \frac{\sum_{j=1}^{m_1} X_{1j}}{m_1} \Big|_{\Theta} \quad \dots \quad X_i = \frac{\sum_{j=1}^{m_i} X_{ij}}{m_i}$$

- $X_{ij} \mid_{\Theta=\theta}$ are iid
- $\mu(\Theta) = E(X_{ij} \mid \Theta)$; $\mu = E\mu(\Theta)$; $v = Var(\mu(\Theta))$
- $v(\Theta) = Var(X_{ij} \mid \Theta)$; $a = E\,v(\Theta)$

The credibility Premium:

$$P_C = Z\bar{X} + (1-Z)\mu$$

$$Z = \frac{m}{m+k}, \quad m = \sum_{i=1}^{n} m_i, \quad k = \frac{a}{v}$$

$$\bar{X} = \frac{m_1 X_1 + m_2 X_2 + \dots + m_n X_n}{m}$$

Example: • In year j, there are N_j claim for
- m_j insurance policies:
- Number of clims for one policy: $N_{ij} \sim Poi(\lambda)$
- $\Lambda \sim Gamma(\alpha, \beta)$

Find the Bulhmann-strub estimate for the number of the claime for ~~per~~ year n+1?

- $\mu(N) = \Lambda$, $\mu = \frac{\alpha}{\beta}$, $v = \frac{\alpha}{\beta^2}$

$V(\Lambda) = \Lambda$, $a = \frac{\alpha}{\beta}$

$k = \frac{a}{v} = \beta$, $Z = \frac{m}{m+\beta}$

estimate: $Z\bar{N} + (1-Z)\frac{\alpha}{\beta}$

Example: • Number of claims per year ~ Poi (λ)

• $\Lambda \sim$ Gamma(6,100)

Year	Number of policies	Number of claimes
1	100 $= m_1$	6
2	150 $= m_2$	8
3	200 $= m_3$	11
4	300	?

One policy: $\frac{m}{m+\beta}\bar{N} + \left(1 - \frac{m}{m+\beta}\right)\frac{\alpha}{\beta}$

$m = 450$

$\bar{N} = \frac{6+8+11}{450} = 0.056$

One policy estimate: $\frac{450}{450+100}(0.056) + \frac{100}{450+100}(0.06)$

$= 0.057$

For 3 policies: $0.057 \times 300 = 17.1$

Example: • Loss per year per exposure $\sim N(\Theta, 1000^2)$

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Θ	Percent of risks
2,000	60%
3,000	30%
4,000	10%

exposure = policy

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	Number of exposure	Aggregate losses
1	24	24 600
2	30	36 000
3	26	28 000

Compute the Bulhmann-Straub Credibility premium for one policy in year 4?

• $m = 24 + 30 + 26 = 80$

• $\mu(\Theta) = \Theta$, $\mu = E(\Theta) = 2500$, $v = Var(\Theta) = 450,000$

$V(\Theta) = 1000^2$, $a = 1000^2$

$$k = \frac{a}{v} = \frac{1000,000}{450,000} = \frac{1}{0.45} = 2.2$$

$$\bar{x} = \frac{24000 + 36000 + 28000}{80} = 1100$$

$$P_C = \frac{80}{80 + 2.2}(1100) + \left(\frac{2.2}{80 + 2.2}\right)(2500) = 1,137.46$$

4.4 Exact Credibility.

Exact credibility is when the premium calculated from Bayesian Credibility and Buhlmann Credibility is the same.

$\pi(\theta)$	$f_{X\|\Theta}(x\|\theta)$	$\pi_{\Theta\|\underline{x}}(\theta\|\underline{x})$
Gamma	Poisson	Gamma
Normal	Normal	Normal
Beta	Bernolli	Beta
Inverse-gama	Exponential	Inverse-gamma

Example: $X|\Theta=\theta \sim \text{Poisson}(\theta)$, $\Theta \sim \text{Gamma}(\alpha,\beta)$

Bayesian Credibility : $X_1, \ldots, X_n$

- model dis: $f_{\underline{X}|\Theta}(\underline{x}|\theta) = e^{-\theta}\frac{\theta^{x_1}}{x_1!} \cdots e^{-\theta}\frac{\theta^{x_n}}{x_n!}$

$$= e^{-n\theta}\frac{\theta^{n\bar{x}}}{\pi\, x_i!}$$

- Joint dis:

$$f_{\underline{X},\Theta}(\underline{x},\theta) \sim e^{-n\theta}\theta^{n\bar{x}}$$
$$\sim \theta^{n\bar{x}+\alpha-1} e^{-(n+\beta)\theta}$$

- Posterior

$$\pi_{\Theta|\underline{x}}(\theta|\underline{x}) \sim \theta^{n\bar{x}+\alpha-1} e^{-(n+\beta)\theta} \sim \text{Gamm}(n\bar{x}+\alpha, n+\beta)$$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha))$

- Premium:

$$E(X_{n+1}|\underline{x}) = \int E(\alpha_{n+1}|\Theta=\theta)\,\pi_{\Theta|\underline{x}}(\theta|\underline{x})\,d\theta$$

$$c = \frac{(n+\beta)^{n\bar{x}+\alpha}}{\Gamma(n\bar{x}+\alpha)} \qquad = c\int \theta\,\theta^{n\bar{x}+\alpha-1} e^{-(n+\beta)\theta}\,d\theta$$

$$= \frac{(n+\beta)^{n\bar{x}+\alpha}}{\Gamma(n\bar{x}+\alpha)} \cdot \frac{\Gamma(n\bar{x}+\alpha+1)}{(n+\beta)^{n\bar{x}+\alpha+1}}$$

$$= \frac{n\bar{x}+\alpha}{n+\beta}$$

- Buhlmann Credibility:

$\mu(\theta) = \theta \longrightarrow \mu = \frac{\alpha}{\beta}$, $\quad v = \frac{\alpha}{\beta^2}$

$v(\theta) = \theta \longrightarrow \quad a = \frac{\alpha}{\beta}$

$k = \frac{a}{v} = \beta$, $\quad Z = \frac{n}{n+\beta}$

- Premium: $\frac{n}{n+\beta}\bar{x} + \frac{\beta}{n+\beta}\frac{\alpha}{\beta} = \frac{n\bar{x}+\alpha}{n+\beta}$

Example: $X|\Theta=\theta \sim N(\theta, 1)$; $\Theta \sim N(\alpha, \beta)$
$x_1, \dots, x_n$

Bayesian Credibility:

model dist.

$$f_{\underline{x}|\theta}(\underline{x}|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-\theta)^2}{2}} \cdots \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_n-\theta)^2}{2}}$$

$$= (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$$

· Joint dis:

$$f_{\underline{X},\Theta}(\underline{x},\Theta) = (2\pi)^{-n/2}\exp\left(-\frac{1}{2}\sum_{i=1}(x_i-\Theta)^2\right)\frac{1}{\sqrt{2\pi\beta}}e^{-\frac{1}{2\beta}(\Theta-\alpha)^2}$$

$$= (2\pi)^{-n/2}(2\pi\beta)^{-1/2}\exp\left[-\frac{1}{2}(n\bar{x}^2-2n\Theta\bar{x}+n\Theta^2)-\frac{1}{2\beta}(\Theta^2-2\alpha\Theta+\alpha^2)\right]$$

$$\simeq \exp\left[-\left(\frac{n}{2}+\frac{1}{2\beta}\right)\Theta^2+\left(n\bar{x}+\frac{\alpha}{\beta}\right)\Theta+\left(\frac{-n\bar{x}^2}{2}-\frac{\alpha^2}{2\beta}\right)\right]$$

$$\simeq \exp\left[\frac{-1}{2\sigma^2}(\Theta^2-2\mu_\Theta\Theta+\mu_\Theta^2)\right]$$

$$2\sigma^2 = \frac{1}{\frac{n}{2}+\frac{1}{2\beta}} \rightarrow \sigma^2 = \sqrt{\frac{1}{n+\frac{1}{\beta}}} = \sqrt{\frac{\beta}{n\beta+1}}$$

$$\frac{\mu_\Theta}{\sigma^2} = n\bar{x}+\frac{\alpha}{\beta} \longrightarrow \mu_\Theta = \left(n\bar{x}+\frac{\alpha}{\beta}\right)\frac{\beta}{n\beta+1}$$

· Poster: $\pi_{\Theta|\underline{x}} \sim N\left(\left(n\bar{x}+\frac{\alpha}{\beta}\right)\frac{\beta}{n\beta+1}, \frac{\beta}{n\beta+1}\right)$

$$E(X_{n+1}|\underline{X}) = \int E(X_{n+1}|\Theta=\theta)\,\pi_{\Theta|\underline{X}}(\theta|x)\,d\theta$$

$$= \int \Theta\,\pi_{\Theta|\underline{x}}(\Theta|\underline{x})\,d\Theta$$

$$= \frac{n\bar{x}\beta+\alpha}{n\beta+1}$$

· Buhlmann credibility:

$\mu(\Theta)=\Theta \rightarrow \mu=\alpha,\ v=\beta$

$V(\Theta)=1 \longrightarrow a=1$

$k=\frac{1}{\beta}$ $\quad Z=\frac{n}{n+\frac{1}{\beta}} = \frac{n\beta}{n\beta+1}$, $\quad P_C = \frac{n\beta}{n\beta+1}\bar{x}+\frac{1}{n\beta+1}\alpha = \frac{n\beta\bar{x}+\alpha}{n\beta+1}$

$F(x) = 1 - (1+\beta x) e^{-\beta x}$

$E(X-d)_+ = EX - E(X \wedge d)$

$E(X \wedge t)$

$= \int_0^t (1 - F(x)) dx = \int_0^t (1+\beta x) e^{-\beta x} dx$

$= \frac{1}{\beta} \int_0^t \beta e^{-\beta x} dx + \frac{1}{\beta} \int_0^t \beta^2 x e^{-\beta x} dx$ Gam$(2, \beta)$

$= \frac{1}{\beta}(1 - e^{-\beta t}) + \frac{1}{\beta}(1 - (1+\beta t) e^{-\beta t})$ $\beta^2 \cdot e^{-\beta x}$

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