Chapter 3

Vectors

Lecture Content

- Coordinate systems
- Vectors and scalar quantities
- Properties of vectors
- Components of vectors and unit vectors

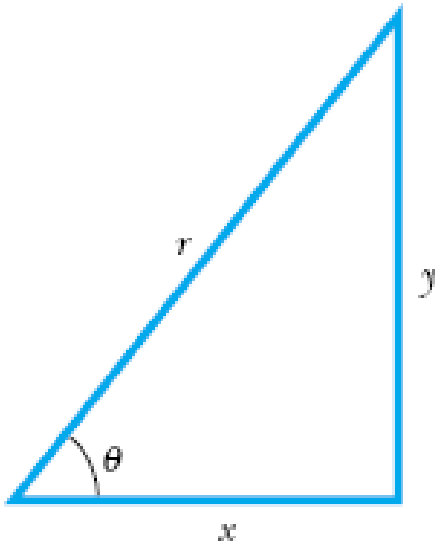
Coordinate systems

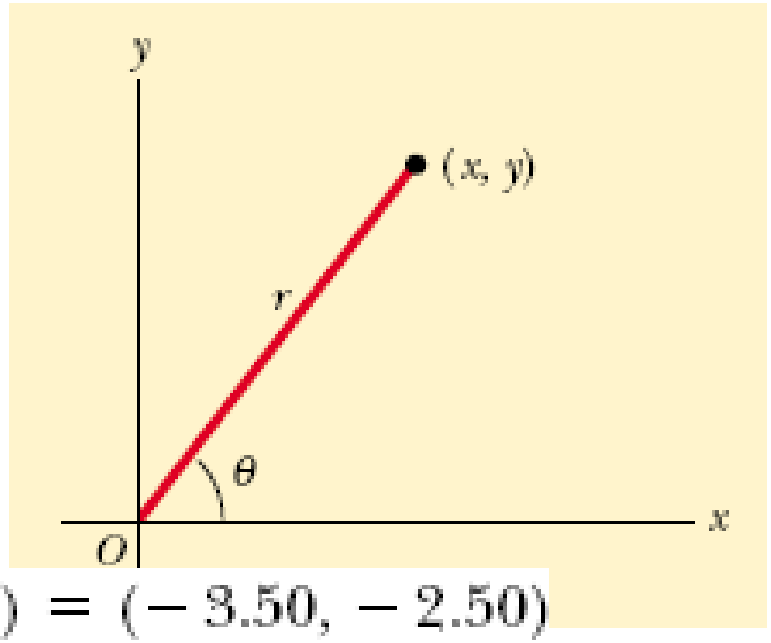
$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

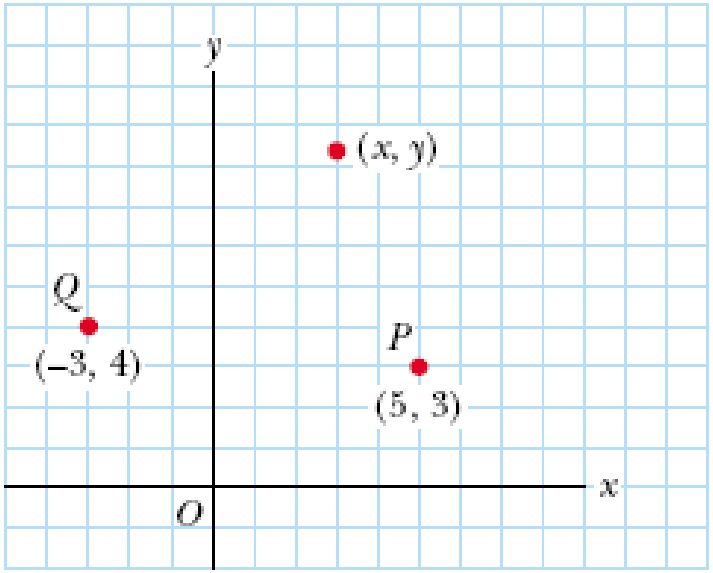
$$\tan\theta = \frac{y}{x}$$

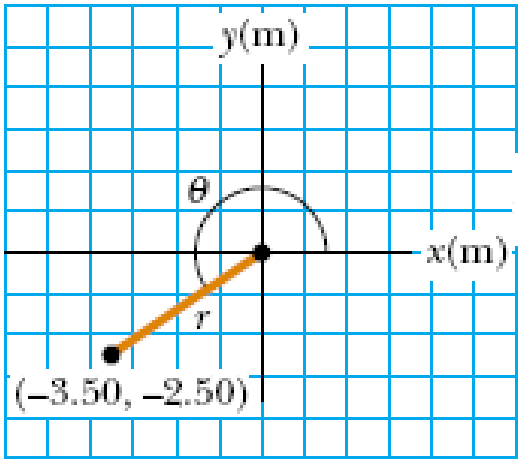
$$r = \sqrt{x^2 + y^2}$$





$$(x, y) = (-3.50, -2.50)$$





$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\ \text{m})^2 + (-2.50\ \text{m})^2}$$

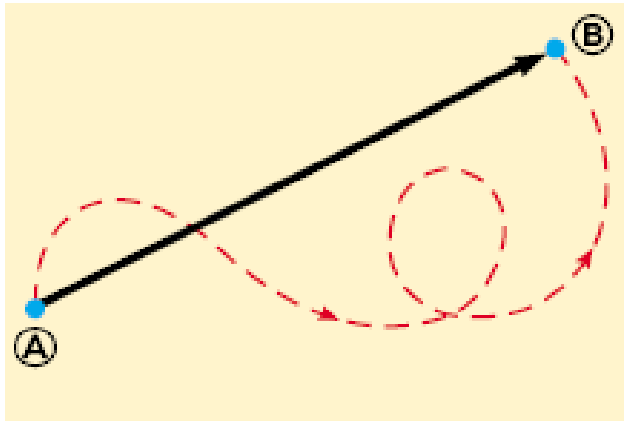
$$4.30\ \text{m}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.50\ \text{m}}{-3.50\ \text{m}} = 0.714$$

$$\theta = 216°$$

Vector and Scalar quantities

- What to dress? Temperature and unit
- Scalar quantity is specified by a single value with appropriate unit and has no direction. Examples: mass, volume, distance
- What you need for flying a Kite? Wind and direction.
- Vector quantity is completely specified by a number and appropriate unit plus direction. Examples: displacement, velocity

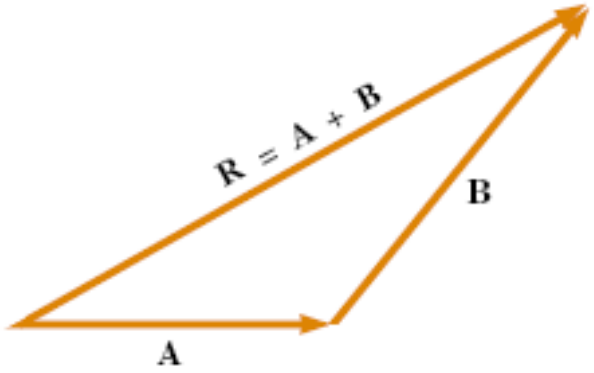


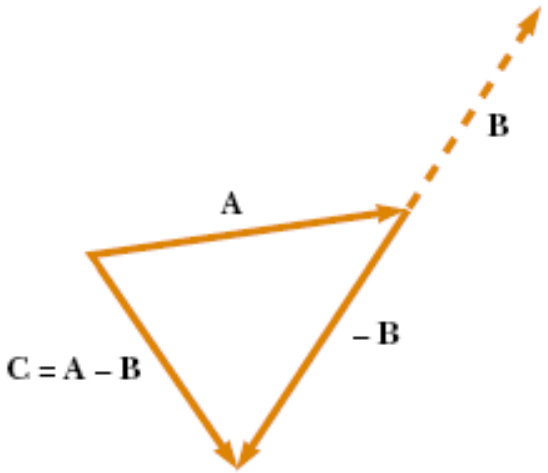
Vectors

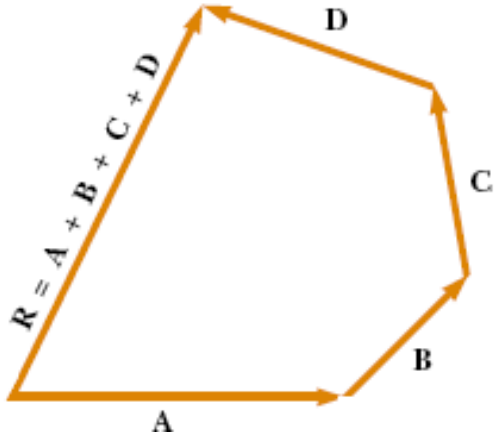
- **A** (bold) is a vector,
- A is a scalar.

$\vec{A}$

- Two vectors are equal if the magnitude and direction are the same.
- Adding vectors
- Negative of a vector
- Vector subtraction







For example, if you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in Figure 3.7, you would find yourself 5.0 m from where you started, measured at an angle of 53° north of east. Your total displacement is the vector sum of the individual displacements.

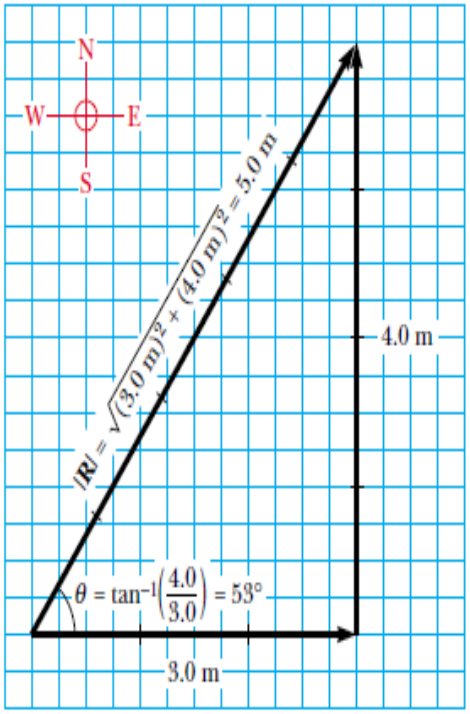


Figure 3.7 Vector addition. Walking first 3.0 m due east and then 4.0 m due north leaves you 5.0 m from your starting point.

When two vectors are added, the sum is independent of the order of the addition. (This fact may seem trivial, but as you will see in Chapter 11, the order is important when vectors are multiplied). This can be seen from the geometric construction in Figure 3.9 and is known as the commutative law of addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

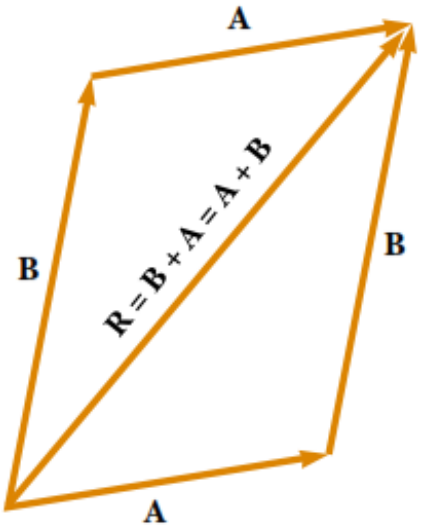


Figure 3.9 This construction shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$—in other words, that vector addition is commutative.

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together. A geometric proof of this rule for three vectors is given in Figure 3.10. This is called the **associative law of addition:**

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{3.6}$$

In summary, **a vector quantity has both magnitude and direction and also obeys the laws of vector addition** as described in Figures 3.6 to 3.10. When two or

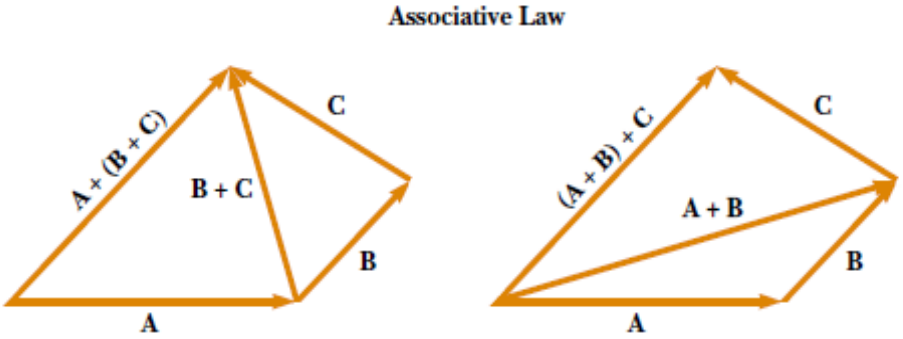


Figure 3.10 Geometric constructions for verifying the associative law of addition.

Negative of a Vector

The negative of the vector **A** is defined as the vector that when added to **A** gives zero for the vector sum. That is, $\mathbf{A} + (-\mathbf{A}) = 0$. The vectors **A** and $-\mathbf{A}$ have the same magnitude but point in opposite directions.

Subtracting Vectors

The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\mathbf{A} - \mathbf{B}$ as vector $-\mathbf{B}$ added to vector $\mathbf{A}$:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{3.7}$$

The geometric construction for subtracting two vectors in this way is illustrated in Figure 3.11a.

Another way of looking at vector subtraction is to note that the difference $\mathbf{A} - \mathbf{B}$ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is what you have to add to the second vector to obtain the first. In this case, the vector $\mathbf{A} - \mathbf{B}$ points from the tip of the second vector to the tip of the first, as Figure 3.11b shows.

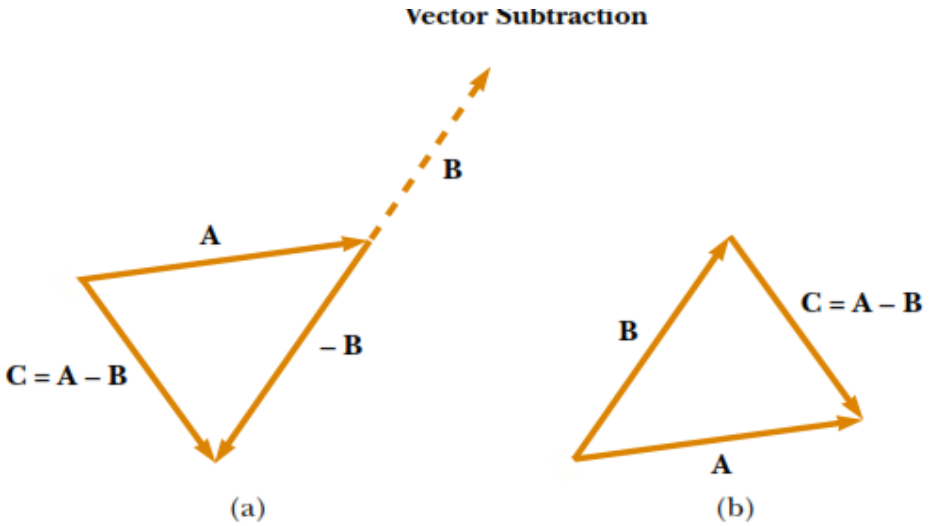


Figure 3.11 (a) This construction shows how to subtract vector **B** from vector **A**. The vector $-\mathbf{B}$ is equal in magnitude to vector **B** and points in the opposite direction. To subtract **B** from **A**, apply the rule of vector addition to the combination of **A** and $-\mathbf{B}$: Draw **A** along some convenient axis, place the tail of $-\mathbf{B}$ at the tip of **A**, and **C** is the difference $\mathbf{A} - \mathbf{B}$. (b) A second way of looking at vector subtraction. The difference vector $\mathbf{C} = \mathbf{A} - \mathbf{B}$ is the vector that we must add to **B** to obtain **A**.

Example 3.2 A Vacation Trip

A car travels 20.0 km due north and then 35.0 km in a direction 60.0° west of north, as shown in Figure 3.12a. Find the magnitude and direction of the car's resultant displacement.

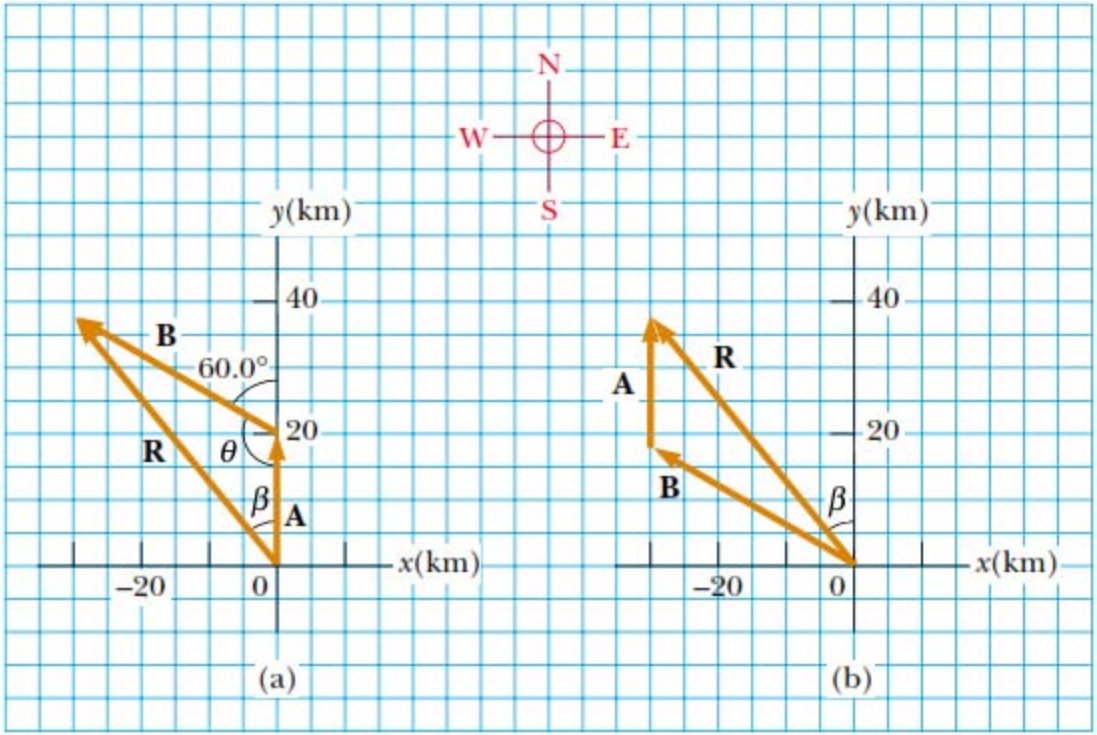


Figure 3.12 (Example 3.2) (a) Graphical method for finding the resultant displacement vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$. (b) Adding the vectors in reverse order ($\mathbf{B} + \mathbf{A}$) gives the same result for $\mathbf{R}$.

The second way to solve the problem is to analyze it algebraically. The magnitude of **R** can be obtained from the law of cosines as applied to the triangle (see Appendix B.4). With $\theta = 180^\circ - 60^\circ = 120^\circ$ and $R^2 = A^2 + B^2 - 2AB\cos\theta$, we find that

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$= \sqrt{(20.0\text{ km})^2 + (35.0\text{ km})^2 - 2(20.0\text{ km})(35.0\text{ km})\cos 120^\circ}$$

$$= \boxed{48.2\text{ km}}$$

$$\frac{\sin\beta}{B} = \frac{\sin\theta}{R}$$

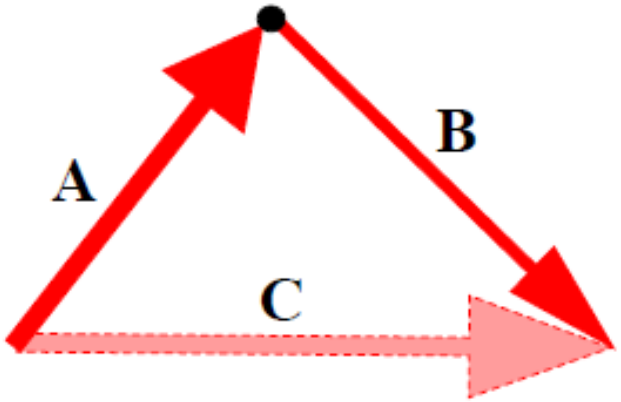
$$\sin\beta = \frac{B}{R}\sin\theta = \frac{35.0\text{ km}}{48.2\text{ km}}\sin 120^\circ = 0.629$$

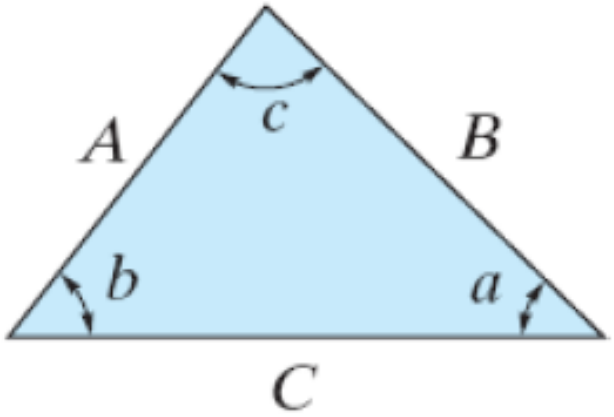
$$\boxed{\beta = 39.0^\circ}$$

The resultant displacement of the car is 48.2 km in a direction 39.0° west of north.

Vectors: Rules of addition

- **Trigonometric Rule**
  - Law of Sines
  - Law of Cosine





Sine law:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

Cosine law:

$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

3.4 Components of a Vector and Unit Vectors

The graphical method of adding vectors is not recommended whenever high accuracy is required or in three-dimensional problems. In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes. These projections are called the **components** of the vector. Any vector can be completely described by its components.

From Figure 3.13 and the definition of sine and cosine, we see that $\cos\theta = A_x/A$ and that $\sin\theta = A_y/A$. Hence, the components of **A** are

$$A_x = A\cos\theta \tag{3.8}$$

$$A_y = A\sin\theta \tag{3.9}$$

These components form two sides of a right triangle with a hypotenuse of length A. Thus, it follows that the magnitude and direction of **A** are related to its components through the expressions

$$A = \sqrt{A_x^2 + A_y^2} \tag{3.10}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3.11}$$

Note that **the signs of the components A_x and A_y depend on the angle θ.** For example, if $\theta = 120°$, then A_x is negative and A_y is positive. If $\theta = 225°$, then both A_x and A_y are negative. Figure 3.14 summarizes the signs of the components when **A** lies in the various quadrants.

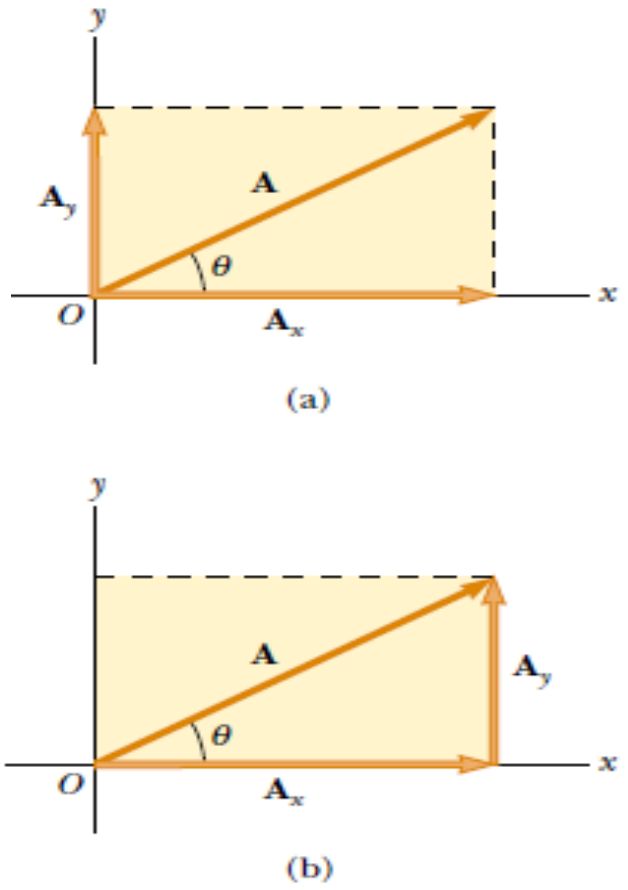


Figure 3.13 (a) A vector **A** lying in the *xy* plane can be represented by its component vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. (b) The *y* component vector $\mathbf{A}_y$ can be moved to the right so that it adds to $\mathbf{A}_x$. The vector sum of the component vectors is **A**. These three vectors form a right triangle.

Figure 3.14 summarizes the signs of the components when A lies in the various quadrants.

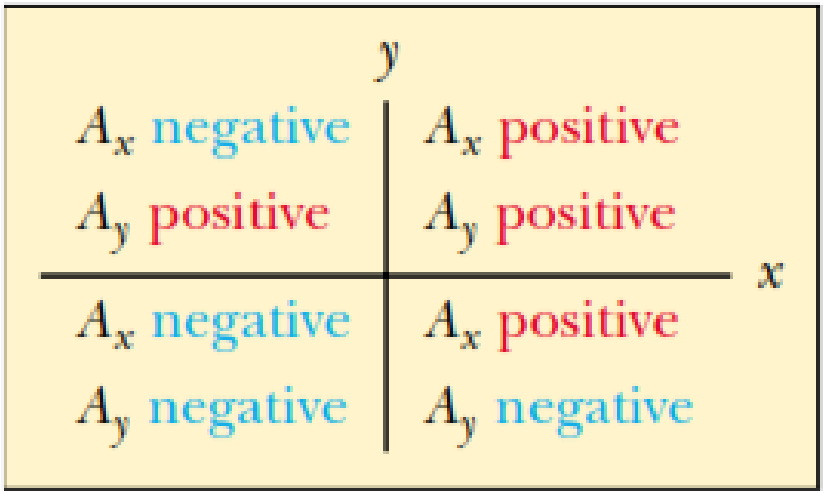
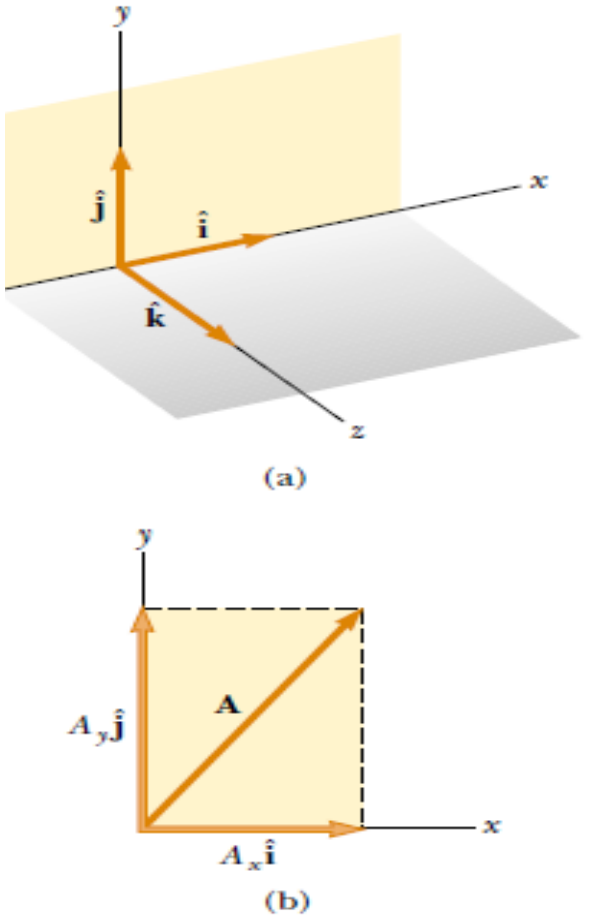


Figure 3.14 The signs of the components of a vector **A** depend on the quadrant in which the vector is located.

Unit Vectors

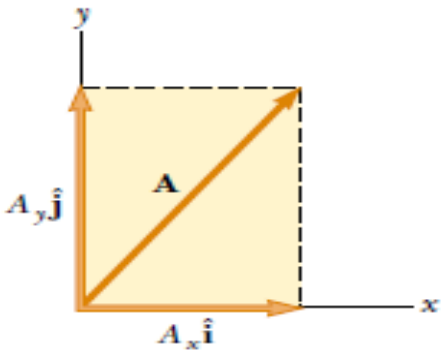
Vector quantities often are expressed in terms of unit vectors. **A unit vector is a dimensionless vector having a magnitude of exactly 1.** Unit vectors are used to specify a given direction and have no other physical significance. They are used solely as a convenience in describing a direction in space. We shall use the symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to represent unit vectors pointing in the positive x, y, and z directions, respectively. (The "hats" on the symbols are a standard notation for unit vectors.) The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ form a set of mutually perpendicular vectors in a right-handed coordinate system, as shown in Figure 3.16a. The magnitude of each unit vector equals 1; that is, $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1$.



Active Figure 3.16 (a) The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are directed along the x, y, and z axes, respectively. (b) Vector $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$ lying in the xy plane has components A_x and A_y.

Consider a vector $\mathbf{A}$ lying in the xy plane, as shown in Figure 3.16b. The product of the component A_x and the unit vector $\hat{\mathbf{i}}$ is the vector $A_x\hat{\mathbf{i}}$, which lies on the x axis and has magnitude $|A_x|$. (The vector $A_x\hat{\mathbf{i}}$ is an alternative representation of vector $\mathbf{A}_x$.) Likewise, $A_y\hat{\mathbf{j}}$ is a vector of magnitude $|A_y|$ lying on the y axis. (Again, vector $A_y\hat{\mathbf{j}}$ is an alternative representation of vector $\mathbf{A}_y$.) Thus, the unit–vector notation for the vector $\mathbf{A}$ is

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \tag{3.12}$$



(b)

Active Figure 3.16 (a) The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are directed along the x, y, and z axes, respectively. (b) Vector $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$ lying in the xy plane has components A_x and A_y.

For example, consider a point lying in the xy plane and having Cartesian coordinates (x, y), as in Figure 3.17. The point can be specified by the **position vector r**, which in unit–vector form is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \tag{3.13}$$

This notation tells us that the components of **r** are the lengths x and y.

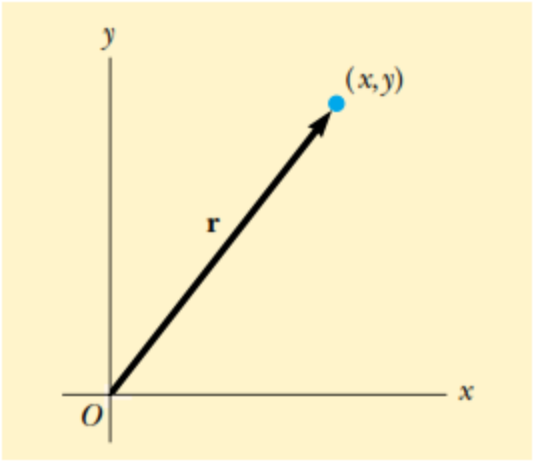


Figure 3.17 The point whose Cartesian coordinates are (x, y) can be represented by the position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

We obtain the magnitude of **R** and the angle it makes with the x axis from its components, using the relationships

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \tag{3.16}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x} \tag{3.17}$$

We can check this addition by components with a geometric construction, as shown in Figure 3.18. Remember that you must note the signs of the components when using either the algebraic or the graphical method.

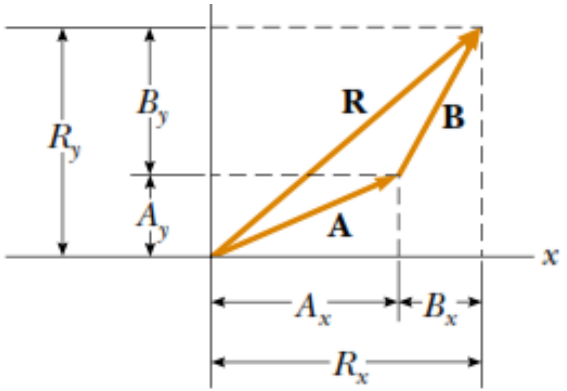


Figure 3.18 This geometric construction for the sum of two vectors shows the relationship between the components of the resultant **R** and the components of the individual vectors.

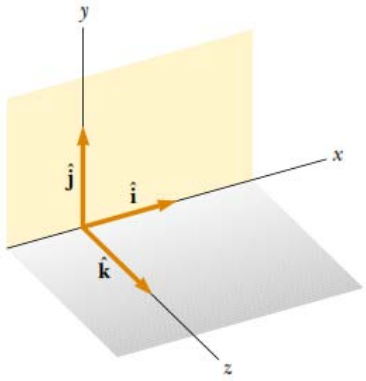
At times, we need to consider situations involving motion in three component directions. The extension of our methods to three-dimensional vectors is straightforward. If **A** and **B** both have x, y, and z components, we express them in the form

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{3.18}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{3.19}$$

The sum of **A** and **B** is

$$\mathbf{R} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}} \tag{3.20}$$



Note that Equation 3.20 differs from Equation 3.14: in Equation 3.20, the resultant vector also has a z component $R_z = A_z + B_z$. If a vector **R** has x, y, and z components, the magnitude of the vector is $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$. The angle θ_x that **R** makes with the x axis is found from the expression $\cos\theta_x = R_x/R$, with similar expressions for the angles with respect to the y and z axes.

Example 3.3 The Sum of Two Vectors

Find the sum of two vectors **A** and **B** lying in the *xy* plane and given by

$$\mathbf{A} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\ \text{m} \qquad \text{and} \qquad \mathbf{B} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\ \text{m}$$

Solution You may wish to draw the vectors to *conceptualize* the situation. We *categorize* this as a simple plug-in problem. Comparing this expression for **A** with the general expression $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$, we see that $A_x = 2.0$ m and $A_y = 2.0$ m. Likewise, $B_x = 2.0$ m and $B_y = -4.0$ m. We obtain the resultant vector **R**, using Equation 3.14:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (2.0 + 2.0)\hat{\mathbf{i}}\ \text{m} + (2.0 - 4.0)\hat{\mathbf{j}}\ \text{m}$$
$$= (4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}})\ \text{m}$$

or

$$R_x = 4.0\ \text{m} \qquad R_y = -2.0\ \text{m}$$

The magnitude of **R** is found using Equation 3.16:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\ \text{m})^2 + (-2.0\ \text{m})^2} = \sqrt{20}\ \text{m}$$
$$= 4.5\ \text{m}$$

We can find the direction of **R** from Equation 3.17:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\ \text{m}}{4.0\ \text{m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the *x* axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\theta = 333°$.

Example 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements: $\mathbf{d}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})$ cm, $\mathbf{d}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})$ cm and $\mathbf{d}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$ cm. Find the components of the resultant displacement and its magnitude.

Solution Three-dimensional displacements are more difficult to imagine than those in two dimensions, because the latter can be drawn on paper. For this problem, let us *conceptualize* that you start with your pencil at the origin of a piece of graph paper on which you have drawn x and y axes. Move your pencil 15 cm to the right along the x axis, then 30 cm upward along the y axis, and then 12 cm *vertically away* from the graph paper. This provides the displacement described by $\mathbf{d}_1$. From this point, move your pencil 23 cm to the right parallel to the x axis, 14 cm parallel to the graph paper in the $-y$ direction, and then 5.0 cm vertically downward toward the graph paper. You are now at the displacement from the origin described by $\mathbf{d}_1 + \mathbf{d}_2$. From this point, move your pencil 13 cm to the left in the $-x$ direction, and (finally!) 15 cm parallel to the graph paper along the y axis. Your final position is at a displacement $\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$ from the origin.

Despite the difficulty in conceptualizing in three dimensions, we can *categorize* this problem as a plug-in problem due to the careful bookkeeping methods that we have developed for vectors. The mathematical manipulation keeps track of this motion along the three perpendicular axes in an organized, compact way:

$$\begin{aligned}\mathbf{R} &= \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3 \\ &= (15 + 23 - 13)\hat{\mathbf{i}}\ \text{cm} + (30 - 14 + 15)\hat{\mathbf{j}}\ \text{cm} \\ &\quad + (12 - 5.0 + 0)\hat{\mathbf{k}}\ \text{cm} \\ &= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\ \text{cm}\end{aligned}$$

The resultant displacement has components $R_x = 25$ cm, $R_y = 31$ cm, and $R_z = 7.0$ cm. Its magnitude is

$$\begin{aligned}R &= \sqrt{R_x^2 + R_y^2 + R_z^2} \\ &= \sqrt{(25\ \text{cm})^2 + (31\ \text{cm})^2 + (7.0\ \text{cm})^2} = 40\ \text{cm}\end{aligned}$$

Example 3.5 Taking a Hike

Interactive

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.

Solution We *conceptualize* the problem by drawing a sketch as in Figure 3.19. If we denote the displacement vectors on the first and second days by **A** and **B**, respectively, and use the car as the origin of coordinates, we obtain the vectors shown in Figure 3.19. Drawing the resultant **R**, we can now *categorize* this as a problem we've solved before—an addition of two vectors. This should give you a hint of the power of categorization—many new problems are very similar to problems that we have already solved if we are careful to conceptualize them.

We will *analyze* this problem by using our new knowledge of vector components. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive x axis. From Equations 3.8 and 3.9, its components are

$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = -17.7\text{ km}$$

The negative value of A_y indicates that the hiker walks in the negative y direction on the first day. The signs of A_x and A_y also are evident from Figure 3.19.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east. Its components are

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = 20.0\text{ km}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = 34.6\text{ km}$$

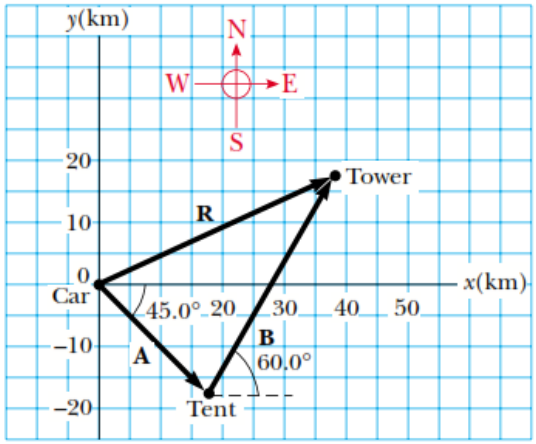


Figure 3.19 (Example 3.5) The total displacement of the hiker is the vector **R** = **A** + **B**.

(B) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

Solution The resultant displacement for the trip $\mathbf{R} = \mathbf{A} + \mathbf{B}$ has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7\ \text{km} + 20.0\ \text{km} = 37.7\ \text{km}$$

$$R_y = A_y + B_y = -17.7\ \text{km} + 34.6\ \text{km} = 16.9\ \text{km}$$

In unit–vector form, we can write the total displacement as

$$\mathbf{R} = (37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\ \text{km}$$

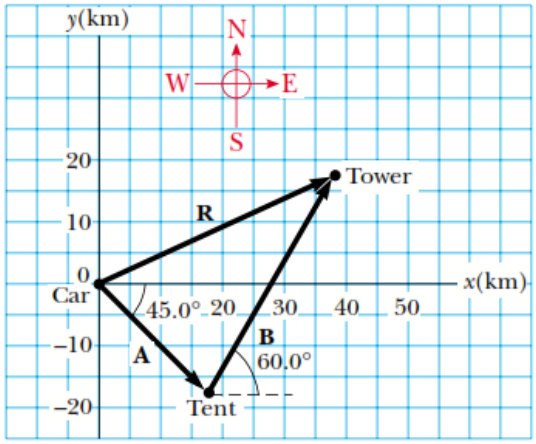
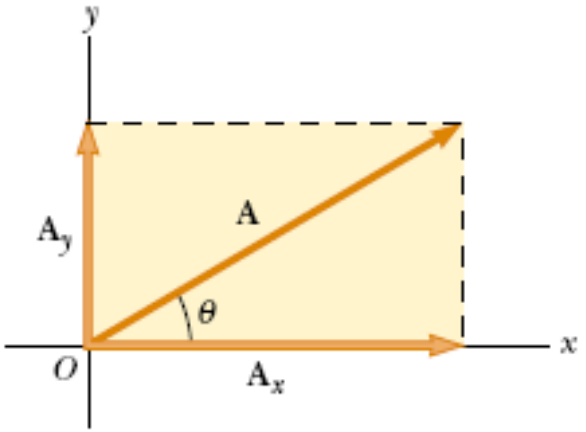


Figure 3.19 (Example 3.5) The total displacement of the hiker is the vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$.

Unit vectors

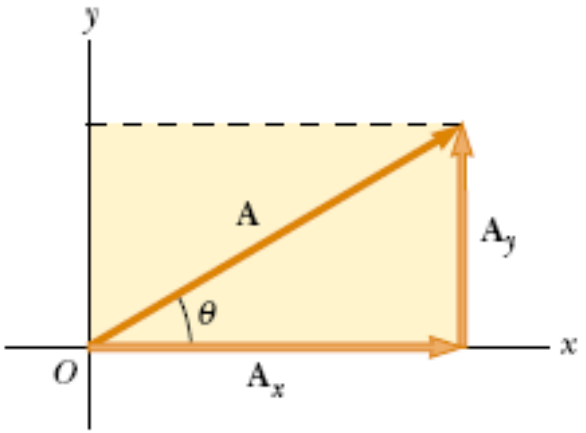
- Multiplying vector by a constant.
- Projection (components) of vector
- Why projection is important? Accuracy, 3D,



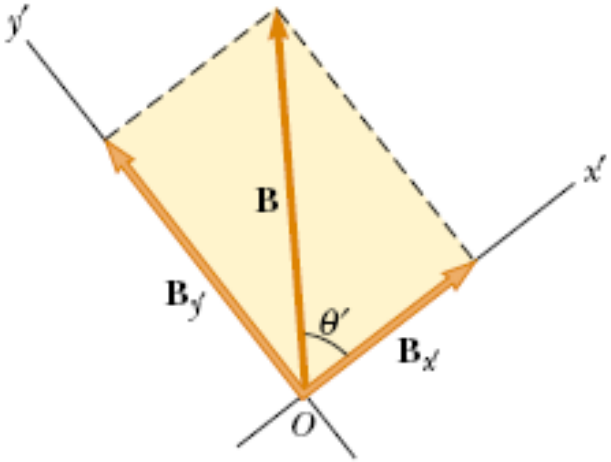
(a)

y	
A_x negative, A_y positive	A_x positive, A_y positive
A_x negative, A_y negative	A_x positive, A_y negative

x

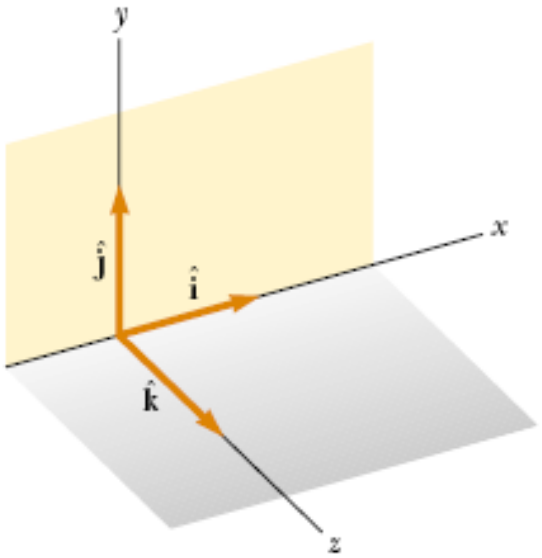


(b)

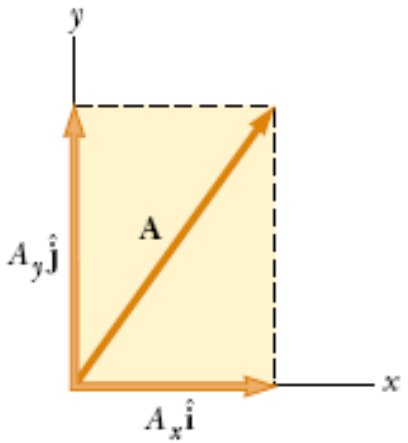


Unit vectors

- Unit vectors in x, y, z coordinate are i, j, k



(a)



(b)

Dot product

- Let **a** and **b** be two vectors defined as

$a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ then

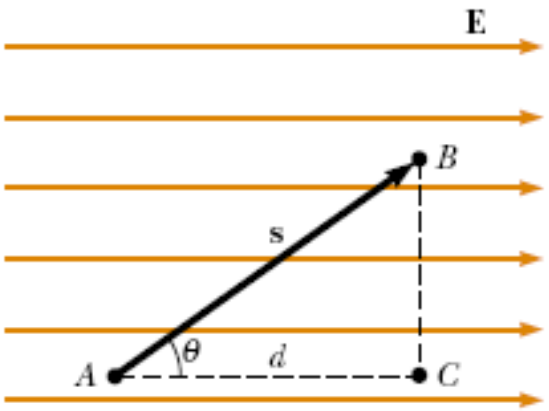
the dot product is a scalar value defined as

$a \cdot b = a_1b_1 + a_2b_2 + a_3 b_3$ Or $= ||a||\,||b|| \cos\theta$

Example 1: 100 N force acting on a block making an angle 60 deg with the horizontal. If the particle moves 2 m horizontally under the action of this force, find the work done by the force? cos 60 = 0.5

Example 2: potential difference

$$V_B - V_A = \Delta V = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$



Cross product

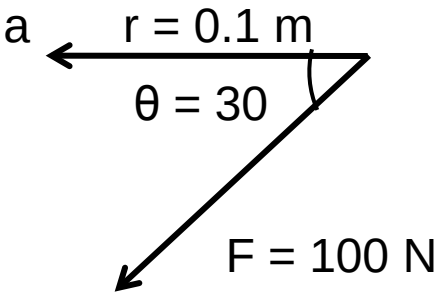
- Let **a** and **b** be two vectors defined as

$a = a_1\, i + a_2\, j + a_3\, k$ and $b = b_1\, i + b_2\, j + b_3\, k$ then the cross product is a **vector** value defined as

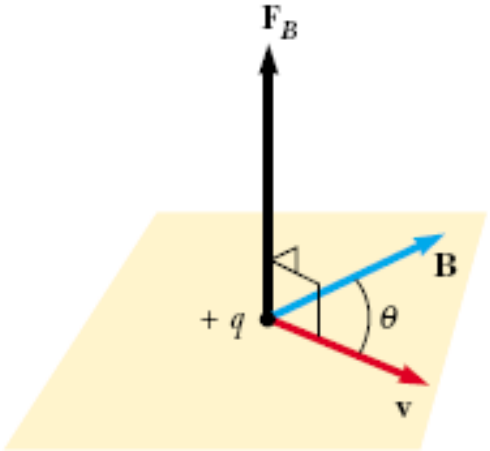
$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\, i + (a_1b_3 - a_3b_1)\, j + (a_1b_2 - a_2b_1)\, k$$

i	j	k
a_1	a_2	a_3
b_1	b_2	b_3

Example 1: 100 N force acting on a wrench (on the horizontal) making an angle 30 deg as shown. Determine the moment acting on the nut at position a. sin 60 = 0.5



Example 2: Magnetic force on a moving charged Particle in a magnetic field

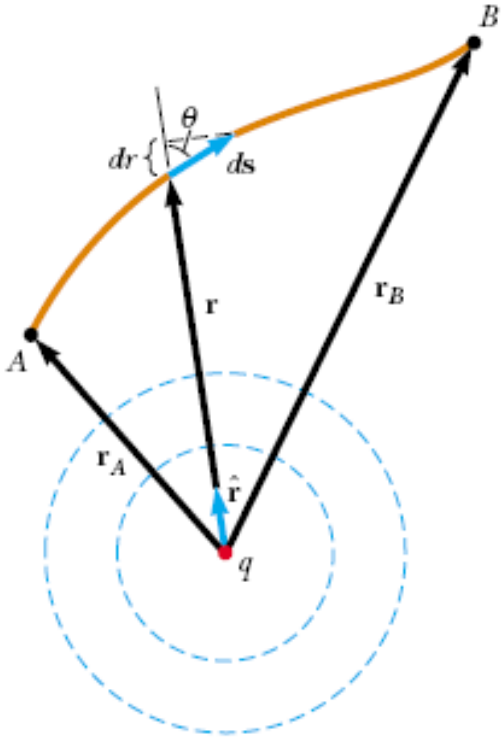


Line Integral

- Integral that is performed along a path between two points.

$$\Delta V = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

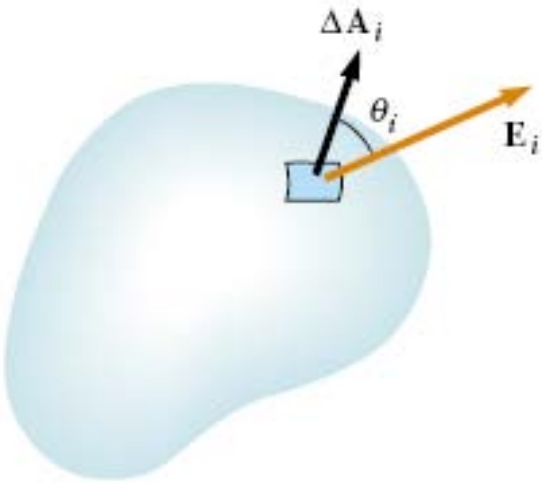
$$\mathbf{E} \cdot d\mathbf{s} = k_e \frac{q}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{s}$$



Surface Integral

- Integral that is performed on a surface.

$$\Phi_E = \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A}$$



- If the surface is closed, the it becomes:

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E_n \, dA$$