Sets

Objectives:

The main purpose for this lesson is to introduce the following:

- ❑ define the set and example.
- ❑ Some concepts of set.
- ❑ The Subset.
- ❑ The Size of a Set.
- ❑ The Power Sets.
- ❑ The Cartesian product

DEFINITION 1

A ***set*** is an unordered collection of objects.

The objects in a set are called the ***elements***, or ***members***, of the set. A set is said to contain its elements. We write **a ∈ A** to denote that a is an element of the set A. The notation **a ∉ A** denotes that a is not an element of the set A.

EXAMPLE 1

The set O of odd positive integers less than 10 can be expressed by $O = \{ l , 3 , 5 , 7, 9\}$.

* This way of describing a set is known as the **roster method.**

EXAMPLE 2

$O = \{x \mid x$ is an odd positive integer less than $10\}$

* Another way to describe a set is to use **set builder notation**. We characterize all those elements in the set by stating the property or properties they must have to be members

or, **specifying the universe as the set** of positive integers, as

$O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10 \}$.

- the set Q^+ of all positive rational numbers can be written as

$Q^+ = \{x \in \mathbb{R} \mid x = \frac{p}{q}$, for some positive integers p and q $\}$.

Some Important Sets

- ℕ= {O, 1 , 2, 3 , . . . } , the set of ***natural numbers***
- ℤ= { . . . , -2, - 1 , 0, 1 , 2, . . . } , the set of ***integers***
- $\mathbb{Z}^{+}$= { I , 2, 3, . . . } , the set of ***positive integers***
- $Q = \{p/q \mid p \in Z, q \in Z, and\ q \neq 0\}$, the set of ***rational numbers***
- ℝ, the set of ***real numbers.***
- ℂ, the set of ***complex numbers.***

Recall the notation for **intervals** of real numbers. When a and b are real numbers with a<b, we write

- $[a, b] = \{x|a \leq x \leq b\}$
- $[a, b) = \{x|a \leq x < b\}$
- $(a, b] = \{x|a < x \leq b\}$
- $(a, b) = \{x|a < x < b\}$

Note that [a, b] is called the closed interval from a to b and (a, b) is called the open interval from a to b.

DEFINITION 2

Two sets are equal if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$

We write A = B if A and B are equal sets.

EXAMPLE 3

The sets { ١ , 3 , 5 } and { 3 , 5 , ١ } are equal, because they have the same elements.

Remarks:

- Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter
- if an element of a set is listed more than once, so

{ ١ , 3 , 3 , 3 , 5 , 5 , 5 , 5 } is the same as the set

{ ١ , 3 , 5 } because they have the same elements.

Some concepts

- **THE EMPTY SET** There is a special set that has no elements. This set is called the empty set,

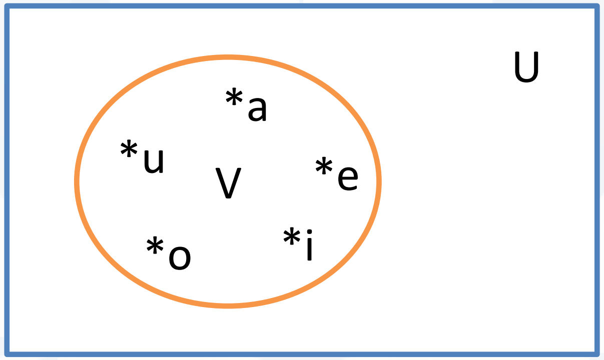
or null set, and is denoted by ∅. The empty set can also be denoted by { }.

- A set with one element is called a **singleton set.**

Venn Diagrams

In Venn diagrams the **universal set U**, which contains all the objects under consideration, is represented by a rectangle.

EXAMPLE 4 Draw a Venn diagram that represents V, the set of vowels in the English alphabet?

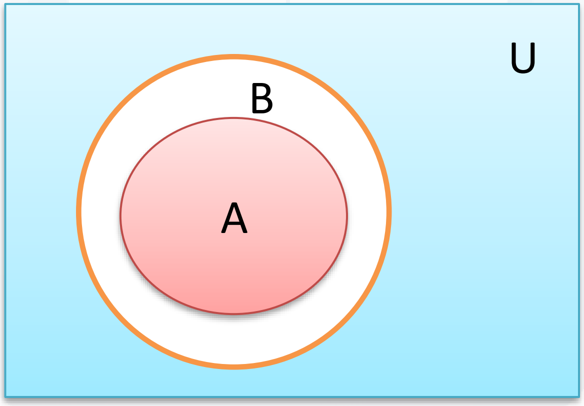


DEFINITION 3

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation **A ⊆ B** to indicate that A is a subset of the set B.

$$\forall x(x \in A \rightarrow x \in B)$$

Note that to show that A is not a subset of B we need only find one element **x ∈ A** with $x \notin B$ **and denoted by** $A \nsubseteq B$



Example 5

(6/147) Suppose that A = {2, 4, 6}, B = {2, 6}, C = {4, 6}, and D = {4, 6, 8}. Determine which of these sets are subsets of which other of these sets.

Solution:

THEOREM 1

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Note : a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is **a proper subset** of B. For $A \subset B$ to be true, it must be the case that

$A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

- Showing Two Sets are Equal :

To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Example:

Sets may have other sets as members. For instance, we have the sets $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$.

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

The Size of a Set

DEFINITION 4

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a ***finite set*** and that n is the ***cardinality of S***. The cardinality of S is denoted by **I S I**.

Example 6

Let A be the set of odd positive integers less than 10. **Then |A| = 5.**

- The null set has no elements, **it follows that**

|Ø| = 0.

DEFINITION 5

A set is said to be **infinite** if it is not finite.

EXAMPLE 7

The set of positive integers $\mathbb{Z}^+$ is infinite.

The Power Set

DEFINITION 6

Given a set S, the ***power set*** of S is the set of all subsets of the set S. The power set of S is denoted by ***P(S).***

EXAMPLE 8

What is the power set of the set {1 , 2} ?

Solution:

P({1,2})=

- What is the power set of the empty set?

- What is the power set of the set {Ø}?

- Note that the empty set and the set itself are members of this set of subsets.

- <u>Remark:</u>

If a set has n elements, then its power set has 2^n elements.

Cartesian Products

DEFINITION 7

The ordered n-tuple $(a_l , a_2 , . . . , a_n)$ is the ordered collection that has a_l as its first element, a_2 as its second element, . . . , and a_n as its nth element.

Equality of two ordered n-tuples

We say that two ordered n -tuples are ***equal*** if and only if each corresponding pair of their elements is equal.

- In other words, $(a_l , a_2 , ... , a_n) = (b_l , b_2 , ... , b_n)$ if and only if $a_i = b_i$, for $i = 1 , 2, ... , n$.
- In particular, 2-tuples are called ordered pairs. The ordered pairs (a,b) and (c,d) are equal if and only if

a =c and b = d.

- Note that (a,b) and (b,a) are not equal unless

a = b.

The Cartesian product of two sets

DEFINITION 8

Let A and B be sets. ***The Cartesian product*** of A and B, denoted by ***A x B***, is the set of all ordered pairs (a,b), where a ϵ A and b ϵ B . Hence,

$$A \; x \; B \; = \; \{(a \, , b) \mid a \; \in \; A \wedge b \; \in \; B \, \}.$$

EXAMPLE 9

What is the Cartesian product of A = { 1 , 2} and B= {a , b, c}?

And Show that the Cartesian product B × A is not equal to the Cartesian product A × B

Solution:

The Cartesian product A x B is

A x B = { (1,a), (1,b) , (1,c), (2,a), (2,b) , (2,c)} .

B × A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}.

Caution!

The Cartesian products A x B and B x A ***are not equal***, unless $A = \emptyset$ or $B = \emptyset$

(so that $A\ x\ B = \emptyset$) or $A = B$

The Cartesian product of sets

DEFINITION 9

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 x\ A_2\ X\ \ldots\ x\ A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

- In other words

$$A_1 X\ A_2\ x\ \cdots\ x\ A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \epsilon A_i \ for\ i = 1, 2, \ldots, n\}.$$

EXAMPLE 10:

What is the Cartesian product A x B x C ,
where A = {0, 1 }, B = { 1 , 2}, and C = {0,1 , 2}?

Solution:

The Cartesian product A x B x C consists of all ordered triples (a , b, c),
where $a \in A, b \in B$, and $c \in C$. Hence,

$$A \; x \; B \; x \; C = \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\}$$

Note:

- We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself.
- Similarly, $A^3 = A \times A \times A$,

$A^4 = A \times A \times A \times A$, and so on.

More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n), | a_i \in A \; for \; i = 1, 2, \ldots, n\}.$$

Homework

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1 (a,b) ,2(a) ,4 ,5 ,7(a,b,d,f) , 9 , 11, 12, 14, 19, 21(a,b), 23, 27, 30, 33(a), 34(b)