Boolean Algebra

Objectives:

The main purpose for this lesson is to introduce the following:

- ❑define Boolean algebra and operations of its.
- ❑Convert Boolean algebra into logical equivalence and convers.
- ❑ define Boolean function and Boolean Expressions.
- ❑ define the dual of a Boolean expression.

Boolean Functions

- In Boolean algebra we work with the set {0,1}, where:

- 0 ≡ F (False)

- 1 ≡ T (True).

The Operations In Boolean Algebra

1. The complementation of an element, denoted with a bar "—" **is defined by:**

$\bar{0} = 1; \qquad \bar{1} = 0$

2. **The sum (+; *OR*):**

1+1=1; 1+0=1; 0+1=1; 0+0=0.

3. **Boolean product(. ; *AND*).**

1.1=1, 1.0=0, 0.1=0, 0.0=0

Translation into a Logical Equivalence

- 0 ≡ F,
- 1 ≡ T ,
- . ≡ Λ,
- + ≡ V,
- —— ≡ ¬

Example1:

Find the value of $1.0 + \overline{(0+1)}$.

Solution:

Using the definition of operations in Boolean algebra, it follows that:

$$
\begin{aligned}
1.0 + \overline{(0+1)} &= 0 + \bar{1} \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

Example 2:

Translate $1 \cdot 0 + \overline{(0 + 1)} = 0$, the equality found in Example 1, into a logical equivalence.

Solution:

Example 3:

Translate the logical equivalence $(T \land T) \lor \neg F \equiv T$ into an identity in Boolean algebra.

Solution:

Boolean Expressions and Boolean Functions

Let $B = \{0,1\}$. Then $B^n=\{(x_1, x_2 , \ldots , x_n) | x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n -tuples of Os and 1s. The variable x is called a Boolean variable if it assumes values only from B , that is, if its only possible values are 0 and 1 .

A function from B^n to B is called a ***Boolean function of degree n***.

Boolean Expressions

Boolean functions can be represented using expressions made up from variables and Boolean operations (., +, $\overline{}$).

The Boolean expressions in the variables x_1 , x_2 , … , x_n are defined recursively as

- 0, 1 , x_1 , x_2 , … , x_n are Boolean expressions;
- if E_1 and E_2 are Boolean expressions, then $\overline{E_1}$, $(E_1.E_2)$, and (E_1+E_2) are Boolean expressions.

EXAMPLE 4

The function $F(x,y) = x\bar{y}$ from the set of ordered pairs of Boolean variables to the set {0,1}

is a ***Boolean function of degree 2*** with

F (1,1)=0, F (1,0) = 1 , F (0,1) = 0, and F(0, 0) = 0.

We display these values of F in Table 1 .

TABLE 1

x	y	$F(x,y)$
1	1	0
1	0	1
0	1	0
0	0	0

EXAMPLE 5

Find the values of the Boolean function represented by $F(x, y, z) = xy + \bar{z}$

Solution:

The values of this function are displayed in Table 2.

TABLE 2

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Identities of Boolean Algebra

EXAMPLE 6

Show that the distributive law **x(y+z)=xy+xz** is valid.

Solution:. The identity holds because the last two columns of the table agree.

TABLE 6 Verifying One of the Distributive Laws.

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

TABLE 5 Boolean Identities.	
Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)x$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

EXAMPLE 7

Translate the distributive law $x+yz=(x+y)(x+z)$ in Table 5 into a logical equivalence.

Solution:

Put $x\equiv p$, $y\equiv q$, & $z\equiv r$, and use the translation of Boolean operations

- $0 \equiv F$,
- $1 \equiv T$,
- $. \equiv \wedge$,
- $+ \equiv \vee$,
- $\overline{} \equiv \neg$

This transforms the Boolean identity into the logical equivalence

$p\vee(q\wedge r)\equiv(p\vee q)\wedge(p \vee r)$.

EXAMPLE 8

Prove the absorption law $x(x+y)=x$ ***using the other identities of Boolean algebra*** shown in

Table 5

Solution:

The steps used to derive this identity and the law used in each step follow:

$x(x+y)=(x+0)(x+y)$	Identity law for the Boolean sum
$= x+0.y$	Distributive law of the Boolean sum over the Boolean product
$= x + Y.0$	Commutative law for the Boolean product
$=x+0$	Domination law for the Boolean product
$= x$	Identity law for the Boolean sum

Duality

- The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging Os and 1 s.
- Duality of a Boolean function F is denoted by F^d

Duality

- 0 interchanged to 1,
- 1 interchanged to 0,
- + interchanged to . ,
- . interchanged to +

EXAMPLE 9

Find the dual of $x(y + 0)$ *and* $\bar{x} \cdot 1 + (\bar{y} + z)$.

Solution:

$x + (y.1)$ *and* $(\bar{x} + 0).(\bar{y}.z)$.

Duality

- 0 interchanged to 1,
- 1 interchanged to 0,
- + interchanged to . ,
- . interchanged to +

Duality Principle

An identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken.

This result, called ***the duality principle***, is useful for obtaining new identities.

EXAMPLE 10

Construct an identity from the absorption law x(x+y)=x ***by taking duals***.

Solution:

Taking the duals of both sides of this identity produces the identity **x+xy = x,** which

is also called an absorption law and is shown in Table 5 .

Duality

- **0 interchanged to 1,**
- **1 interchanged to 0,**
- **+ interchanged to . ,**
- **. interchanged to +**

Homework

Page 818

- 1,
- 3
- 5(a,b),
- 9
- 11,
- 21
- 28(a,d).