Set Operations

Objectives:

The main purpose for this lesson is to introduce the following:

- ❑ define the operations of sets and examples.
- ❑ Some concepts of set.
- ❑ Computer Representation of Sets

The Union

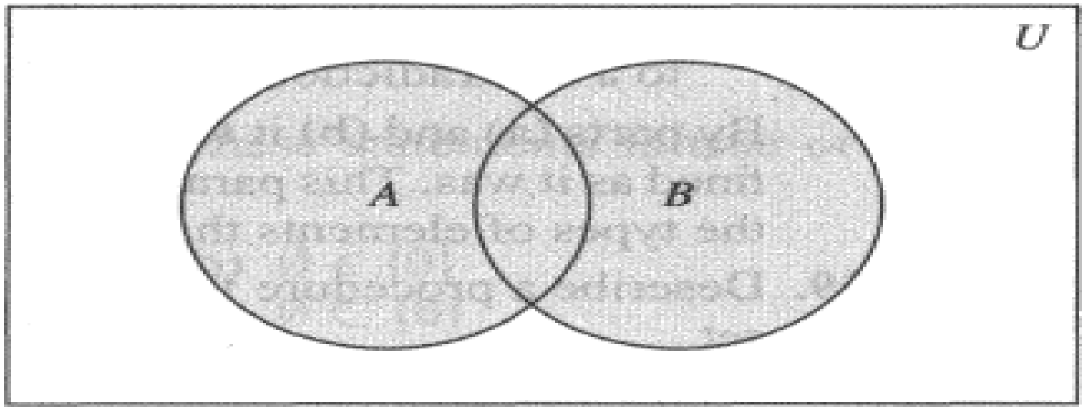
DEFINITION 1

- Let A and B be sets. ***The union*** of the sets A and B , denoted by ***A U B*** , is the set that contains

those elements that are either in A ***or*** in B , ***or*** in both.

- An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B .
- This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram Representing the Union of A and B.

EXAMPLE 1

The ***union*** of the sets {1,3,5} and {1,2,3} is the set {1,2,3,5} ; that is,

{1,3,5} **U** {1,2,3} = {1,2,3,5} .

The Intersection

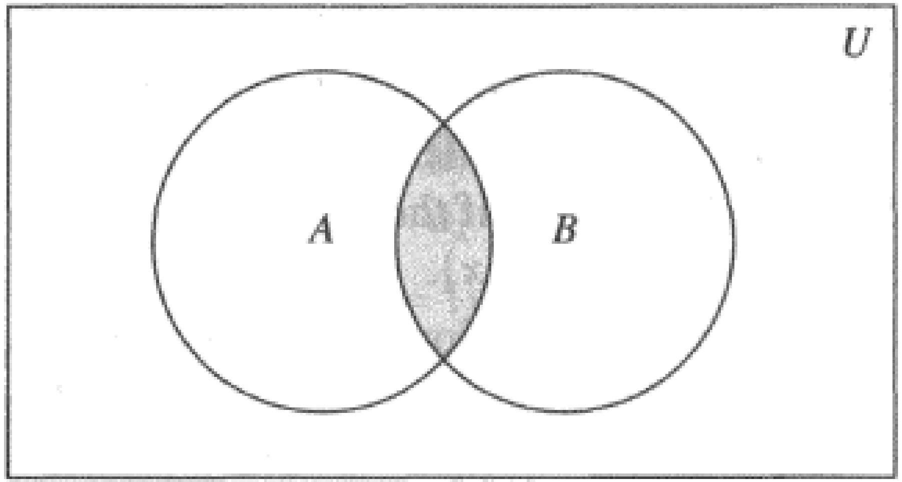
DEFINITION 2

Let A and B be sets. ***The intersection*** of the sets A and B , denoted by $A \cap B$, is the set

containing those elements in both A **and** B .

- An element x belongs to the intersection of the sets A and B if and only if x belongs to A **and** x belongs to B .
- This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\} .$$



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram Representing the Intersection of A and B.

EXAMPLE 2

The intersection of the sets {1,3,5} and {1,2,3} is the set {1,3} ; that is, $\{1,3,5\} \cap \{1,2,3\} = \{1,3\}$.

Disjoint Sets

DEFINITION 3

Two sets are called **disjoint** if their intersection is the empty set.

EXAMPLE 3

Let A = {1,3,5,7,9} and B = {2, 4, 6, 8 , 10} . Because $A \cap B = \emptyset$, ***A and B are disjoint***.

The Cardinality Of a Union Of Two Finite Sets

Note that I A I + I B I counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from IAI+ IBI , elements in A∩B will be counted only once.

Hence, $|A \cup B| = |A| + |B| - |A \cap B|$.

- The generalization of this result to unions of an arbitrary number of sets is called ***the principle of inclusion-exclusion.***

The Difference Of Two Sets

DEFINITION 4

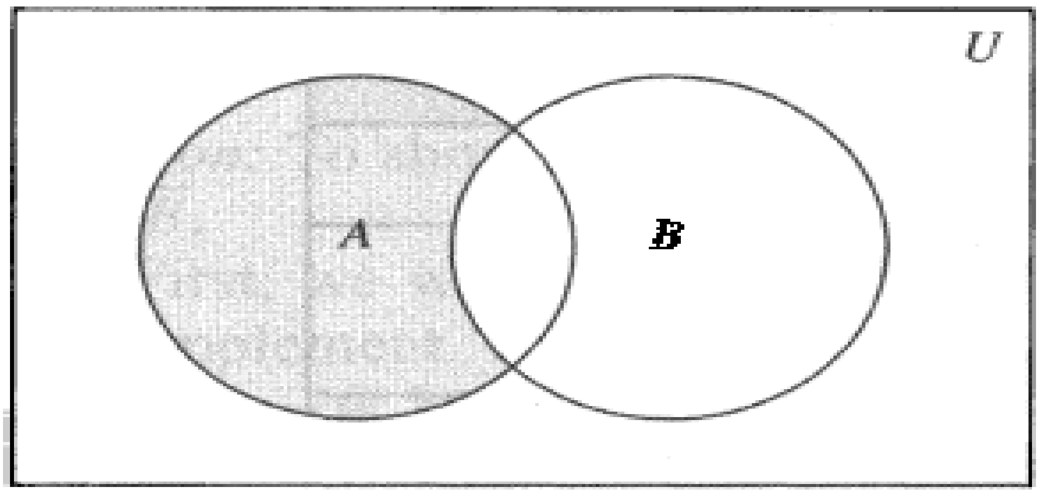
- Let A and B be sets. ***The difference*** of A and B , denoted by ***A - B***, is the set containing those

elements that are in A but not in B .

- The difference of A and B is also called the ***complement of B with respect to A.***
- An element x belongs to *the difference of A and B* if and only if **x ϵ A and x ∉ B** . This tells us that

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

Remark: The difference of sets A and B is sometimes denoted by A\B.



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.

EXAMPLE 4

The difference of {1,3,5} and {1,2,3} is the set {5}; that is, {1,3,5} - {1,2,3} = {5} .

Caution!

This is different from the difference of {1,2,3} and {1,3,5} , which is the set {2}.

{1,2,3}- {1,3,5} = {2}.

The Complement Of a Set

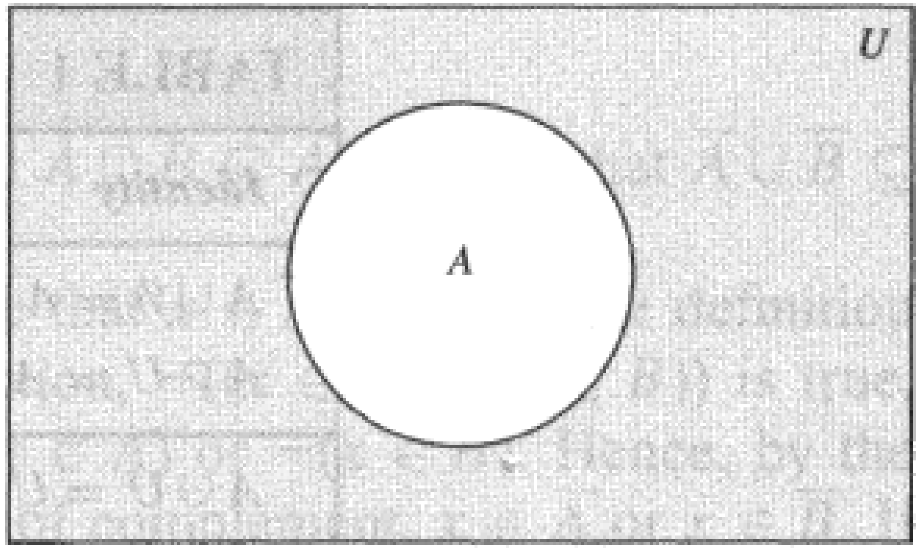
DEFINITION 5

- Let U be the universal set. ***The complement*** of the set A , denoted by $\bar{A}$, is the complement of A with respect to U.
- In other words, the complement of the set A is

$\bar{A} = U - A.$

- An element belongs to $\bar{A}$ if and only if x $\notin$ A . This tells us that

$$\bar{A} = \{x \mid x \notin A\}.$$



$\overline{A}$ is shaded

FIGURE 4 Venn Diagram for the Complement of the Set *A*.

Example 5 (3/136)

Let A = {1, 2, 3, 4, 5} and B = {0, 3, 6}. Find

c) A-B   d) B-A

Solution:

c) A-B =

d) B-A=

Remark:

It is left to the reader to show that we can express the difference of A and B as the intersection of A and the complement of B.

That is,

$$A - B = A \cap \bar{B}.$$

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Example: Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Example:

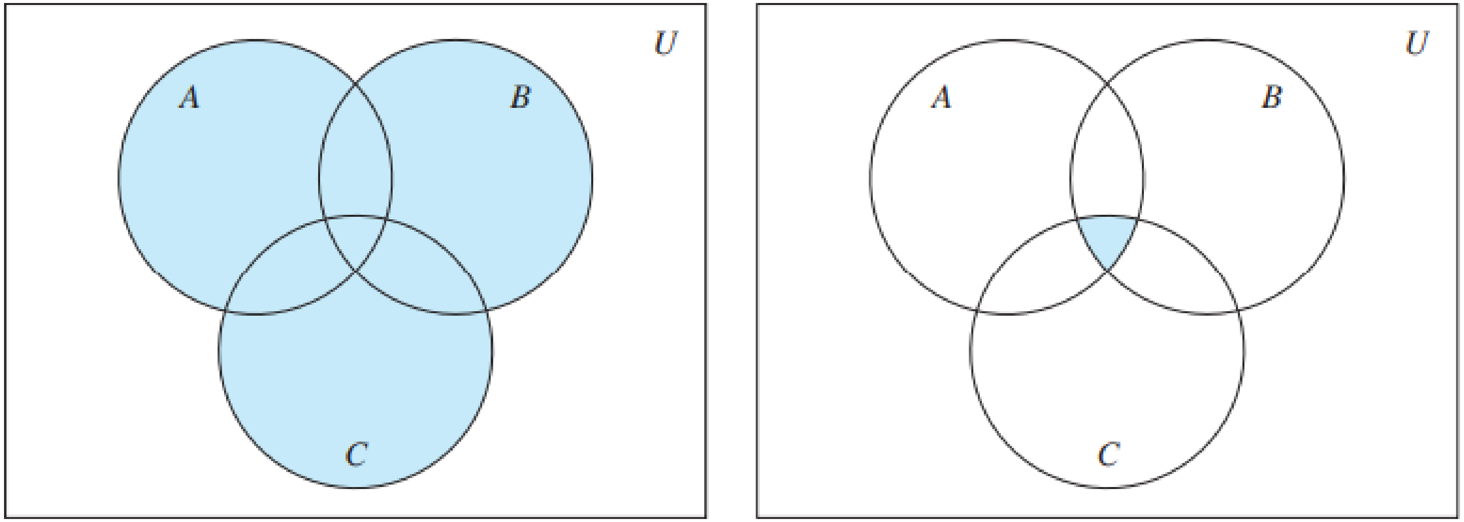
Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \left(\overline{B \cap C}\right) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Generalized Unions and Intersections



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

FIGURE 5 **The Union and Intersection of A, B, and C.**

DEFINITION 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

DEFINITION 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$

Example:

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Computer Representation of Sets

- Assume that the universal set U is finite.
- First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \dots, a_n$.
- Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example:

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

- Represent the subset of All odd integers $\{1,3,5,7,9\}$ in U by 10 1010 1010.
- we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

- The set of all integers in U that do not exceed 5, namely, {1, 2, 3, 4, 5}, is represented by the string 11 1110 0000.

Notes:

- Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1,(see example 19 in book).

What about union and intersection ?

Example:

The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set {1, 2, 3, 4, 5, 7, 9}. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set {1, 3, 5}. ◀

HomeWorks

Page 136/137

- 3
- 5
- 14
- 19
- 25
- 27
- 29
- 52,53
- 59