PHYS-454

The algebraic method for the quantum SHO

Dirac formalism: a new way for representing wavefunctions-a

- According to Dirac, any quantum state ψ is represented by two vectors: The first is a column vector, is denoted as $|\psi\rangle$, and is called ***ket vector***. The second is a row vector and is denoted by $\langle\psi|$ and is called ***bra vector***. These names come from the english word bracket because in this formalism the dot product of two states ψ and ϕ is given by

$$(\psi,\phi)=\langle\psi|\varphi\rangle$$

Dirac formalism: a new way for representing wavefunctions-b

- With this formalism the average value of a physical quantity on a state ψ is denoted by:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(A\psi(x)\right)dx = \langle \psi | A | \psi \rangle$$

- The two vectors are related by the following relations

$$\left(|\psi\rangle\right)^\dagger = \langle \psi|, \quad \left(\langle \psi|\right)^\dagger = |\psi\rangle$$

$$\left(c_1|\psi_1\rangle + c_2|\psi_2\rangle\right)^\dagger = c_1^*\langle \psi_1| + c_2^*\langle \psi_2|$$

…the quantum SHO…

- Since the quantum SHO has equidistant energy eigenvalues which are produced by the ground state energy by "ascending" at a constant step, it is reasonable to think if we could do the same for the SHO eigenfunctions.

…the algebraic solution-a…

- Consider the ket notation for the SHO eigenfunctions:

$$\psi_n(x) \rightarrow |n\rangle$$

- Consider also the following raising and lowering operators a and $a^\dagger$

These operators act as follows on a certain eigenstate of the SHO

…the algebraic solution-b…

$$a^{\dagger}\left|n\right\rangle \rightarrow \left|n+1\right\rangle \qquad a\left|n\right\rangle \rightarrow \left|n-1\right\rangle$$

- This means that the first operator shifts the SHO to the next higher eigenstate, while the second one shifts the SHO to the previous lower eigenstate!

- *Question: what is the result of the action of the operator* $N = a^{\dagger}a$ *on a SHO eigenstate?*

...the algebraic solution-c...

- We define the raising and lowering operators as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \qquad a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

where x and p are the position and momentum operators. With the help of these operators the Hamiltonian takes the form

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)$$

Question: Express the SHO Hamiltonian as a "function" of the raising and lowering operators.

…the algebraic solution-d…

Prove the following relations

$$\left[a,\ a^{\dagger}\right]=1,\qquad \left[N,\ a\right]=-a,\qquad \left[N,\ a^{\dagger}\right]=a^{\dagger}$$

- The most characteristic property of the energy spectrum of the SHO is the equal distance between successive energy eigenvalues.
- *Question: Prove the characteristic property of the SHO energy spectrum property. Find the energy eigenvalues of SHO.*

...the algebraic solution-e...

- The raising and lowering operators are known as creation and destruction operators respectively since the first "creates" a quantum of energy $\hbar\omega$ and thus raises the SHO to the next higher state, while the second "destroys" a quantum of energy $\hbar\omega$ and thus brings the SHO down to the next lower state.
- The operator $N = a^{\dagger}a$ is known a number operator since it gives the number of energy quant in a state $|n\rangle$

…the algebraic solution-f…

- We will prove that the proper forms of the raising and lowering operators are:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

- With them one can built any state $|n\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n}|0\rangle$$

from the vacuum state $|0\rangle$ (n=0) .

…an interesting theorem…

- If for an operator H we can find an operator A for which, $\left[H, A\right] = \xi A$ then:
  a) Operator H has equidistant eigenergies
  b) If $\xi < 0$, operator A is a lowering operator, while if $\xi > 0$, operator A is a raising operator