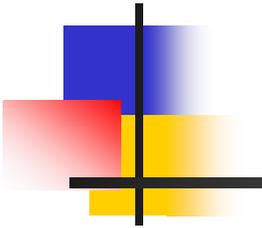
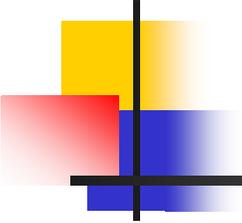


PHYS-454

The algebraic method for the quantum SHO

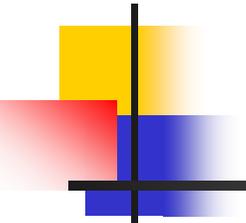




Dirac formalism: a new way for representing wavefunctions-a

- According to Dirac, any quantum state ψ is represented by two vectors: The first is a column vector, is denoted as $|\psi\rangle$, and is called *ket vector*. The second is a row vector and is denoted by $\langle\psi|$ and is called *bra vector*. These names come from the english word bracket because in this formalism the dot product of two states ψ and ϕ is given by

$$(\psi, \phi) = \langle\psi|\phi\rangle$$



Dirac formalism: a new way for representing wavefunctions-b

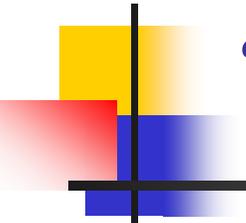
- With this formalism the average value of a physical quantity on a state ψ is denoted by:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x) (A\psi(x)) dx = \langle \psi | A | \psi \rangle$$

- The two vectors are related by the following relations

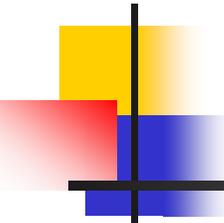
$$(|\psi\rangle)^\dagger = \langle \psi|, \quad (\langle \psi|)^\dagger = |\psi\rangle$$

$$(c_1|\psi_1\rangle + c_2|\psi_2\rangle)^\dagger = c_1^* \langle \psi_1| + c_2^* \langle \psi_2|$$



...the quantum SHO...

- Since the quantum SHO has equidistant energy eigenvalues which are produced by the ground state energy by “ascending” at a constant step, it is reasonable to think if we could do the same for the SHO eigenfunctions.



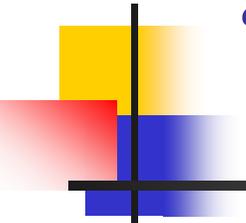
...the algebraic solution-a...

- Consider the ket notation for the SHO eigenfunctions:

$$\psi_n(x) \rightarrow |n\rangle$$

- Consider also the following raising and lowering operators a and a^\dagger

These operators act as follows on a certain eigenstate of the SHO



...the algebraic solution-b...

$$a^\dagger |n\rangle \rightarrow |n+1\rangle \quad a |n\rangle \rightarrow |n-1\rangle$$

- This means that the first operator shifts the SHO to the next higher eigenstate, while the second one shifts the SHO to the previous lower eigenstate!
- *Question: what is the result of the action of the operator $N = a^\dagger a$ on a SHO eigenstate?*

...the algebraic solution-c...

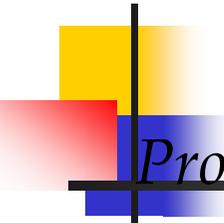
- We define the raising and lowering operators as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right)$$

where x and p are the position and momentum operators. With the help of these operators the Hamiltonian takes the form

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

Question: Express the SHO Hamiltonian as a “function” of the raising and lowering operators.



...the algebraic solution-d...

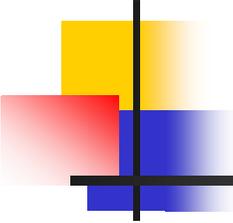
Prove the following relations

$$\left[a, a^\dagger \right] = 1, \quad \left[N, a \right] = -a, \quad \left[N, a^\dagger \right] = a^\dagger$$

- The most characteristic property of the energy spectrum of the SHO is the equal distance between successive energy eigenvalues.
- *Question: Prove the characteristic property of the SHO energy spectrum property. Find the energy eigenvalues of SHO.*

...the algebraic solution-e...

- The raising and lowering operators are known as creation and destruction operators respectively since the first “creates” a quantum of energy $\hbar\omega$ and thus raises the SHO to the next higher state, while the second “destroys” a quantum of energy $\hbar\omega$ and thus brings the SHO down to the next lower state.
- The operator $N = a^\dagger a$ is known a number operator since it gives the number of energy quant in a state $|n\rangle$



...the algebraic solution-f...

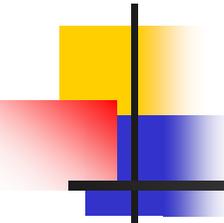
- We will prove that the proper forms of the raising and lowering operators are:

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

- With them one can built any state $|n\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

from the vacuum state $|0\rangle$ ($n=0$) .



...an interesting theorem...

- If for an operator H we can find an operator A for which, $[H, A] = \xi A$ then:
 - a) Operator H has equidistant eigenenergies
 - b) If $\xi < 0$, operator A is a lowering operator, while if $\xi > 0$, operator A is a raising operator