PHYS-454

The position and momentum representations

The continuous spectrum-a

- So far we have seen problems where the involved operators have a discrete spectrum of eigenfunctions and eigenvalues.
- What happens if the spectrum of an operator A is continuous?
- The most important property in this case are that the corresponding wavefunctions **are not square integrable!**

The continuous spectrum-b

- And the questions arises naturally:
- Can such eigenfunctions be accepatable from the moment that they correspond to infinite probability?
- To answer and clarify this topic we make the following straightforward correspondence:

$$\psi = \sum c_n \psi_n \rightarrow \psi \equiv \int c(a)\psi_a(x)\,dx$$

$$A\psi_a(x) = a\psi_a(x)$$

The continuous spectrum-c

- Similarly:

$$c(a) = (\psi_a, \psi) \qquad\qquad P(a) = |c(a)|^2$$

- Now there is the question: What is the property which corresponds to the normalization relation for the discrete spectrum $(\psi_n, \psi_m) = \delta_{nm}$?
- The answer is the so called **Dirac Delta Function**.

$$(\psi_{a'}, \psi_a) = \delta(a' - a)$$

Properties of the Dirac function

$$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$$

$$\int_{-\infty}^{+\infty} \delta(x-a)\phi(x)\,dx = \phi(a)$$

$$x\delta(x) = 0$$

$$\delta(\lambda x) = \frac{1}{|\lambda|}\delta(x)$$

$$\delta(-x) = \delta(x)$$

$$\int_{-\infty}^{+\infty} e^{ikx}\,dx = 2\pi\delta(k)$$

The continuous spectrum-d

- With the introduction of delta function we can establish the following properties and complete this discussion:

$$\int_{-\infty}^{+\infty} \left| c(a) \right|^2 da = 1$$

Different representations in QM

- So far we have talked about wavefunctions which depend on the position: $\psi(x)$.
- How this happened? Why we chose position and not another variable?
- In quantum mechanics we can choose any base we wish, provided that it is made up from the eigenvectors of a relevant physical quantity.

The position representation and the ket formalism-a

- In ket formalism all the above relations are given as follows:

$$x\left|x'\right\rangle = x'\left|x'\right\rangle \qquad \left\langle x''\middle|x'\right\rangle = \delta\left(x'' - x'\right)$$

$$\left|a\right\rangle = \int dx' \left|x'\right\rangle\left\langle x'\middle|a\right\rangle$$

- The expansion coefficient $\left\langle x'\middle|a\right\rangle$ is interpreted in such a way that $\left|\left\langle x'\middle|a\right\rangle\right|^2 dx'$ is the probability for the particle to be found in a narrow interval dx' around x'.

The position representation and the ket formalism-b

- In our formalism the inner product $\langle x' | a \rangle$ is what is usually referred to as the **wave function** $\psi_a(x')$ for the state $|a\rangle$

$$\langle x' | a \rangle = \psi_a(x')$$

- Consider now the inner product between two states $\langle \beta | a \rangle$, using the completeness of $|x'\rangle$, we have:

$$\int |x'\rangle\langle x'| dx = \mathbf{1}$$

$$\langle \beta | a \rangle = \int dx' \langle \beta | x' \rangle \langle x' | a \rangle = \int dx' \psi_\beta^*(x') \psi_a(x')$$

The momentum representation and the ket formalism-c

- So $\langle \beta | a \rangle$ characterizes the overlap between the two wavefunctions. Note that we do not define $\langle \beta | a \rangle$ as the overlap integral: this **follows** from the completeness postulate for $|x'\rangle$. The more general interpretation of $\langle \beta | a \rangle$, **independent of representations**, is that it represents the probability amplitude for state $| a >$ to be found in state $| \beta >$.

Example: Interpret the expansion $| a > = \Sigma | a' >< a' | a >$ using the language of wave functions.

THE MOMENTUM REPRESENTATION:

- As we saw for the position (in ket formalism) the eigenvalue equation is:

$$\hat{x}\left|x\right\rangle = x\left|x\right\rangle$$

- In the continuous basis formed by the position eigenvectors $\left|x\right\rangle,\ x \in (-\infty, \infty)$ the arbitrary ket $\left|\psi\right\rangle$ will be represented by its coordinates $c_x = c(x)$ given by $c(x) = \left\langle x \middle| \psi \right\rangle$ which is the **probability amplitude of position** which is the familiar wavefunction:

$$\psi(x) = \left\langle x \middle| \psi \right\rangle \qquad \int \left|x\right\rangle\left\langle x\right| dx = \mathbf{1}$$

…$\psi(x)$ is the representation of the arbitrary ket $|\psi>$ in the basis of the position eigenvectors..

The momentum representation and the ket formalism-a.

- In ket formalism all the above relations are given as follows:

$$p|p'\rangle = p'|p'\rangle \quad \langle p''|p'\rangle = \delta(p'' - p')$$

$$|a\rangle = \int dp' |p'\rangle\langle p'|a\rangle \qquad \int |p'\rangle\langle p'| dp = \mathbf{1}$$

- The expansion coefficient $\langle p'|a\rangle$ is interpreted in such a way that $|\langle p'|a\rangle|^2 dp'$ is the probability for the particle to be found in a narrow interval dp' around p'.

The momentum representation and the ket formalism-b.

- In our formalism the inner product $\langle p' | a \rangle$ is what is usually referred to as the **wave function** $\tilde{\psi}_a(p')$ for the state $|a\rangle$

$$\langle p' | a \rangle = \tilde{\psi}_a(p')$$

- If $|a>$ is normalized then

$$\int dp' \langle a | p' \rangle \langle p' | a \rangle = \int dp' \left| \tilde{\psi}_a(p') \right|^2 = 1$$

Connection between the two representations-a.

- The quantity $<x' \mid p'>$ is called the **transformation function** from x-representation to the p-representation. From the position representation of the momentum operator we get:

$$p'\left\langle x' \middle| p' \right\rangle = -i\hbar \frac{\partial}{\partial x'} \left\langle x' \middle| p' \right\rangle \Rightarrow \left\langle x' \middle| p' \right\rangle = N \exp\left(\frac{ip'x'}{\hbar} \right)$$

Even though the transformation function is a function of two variables x' and p', we can temporarily regard it as a function of x' with p' fixed.

Connection between the two representations-b.

- It can be viewed as the probaility amplitude for the momentum eigenstate specified by p' to be found at position x'; in other words, it is just the wavefunction for the momentum eigenstate $|p'>$.
- It is obvious that it is a plane wave.
- It can be shown that the normalization constant is given by:

$$N = 1/\sqrt{2\pi\hbar}$$

Connection between the two representations-c

- Now we can demonstrate the connection between the two representations:

$$\langle x' | a \rangle = \int dp' \langle x' | p' \rangle \langle p' | a \rangle \qquad \langle p' | a \rangle = \int dx' \langle p' | x' \rangle \langle x' | a \rangle$$

$$\psi_a(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \exp\left(\frac{ip'x^{;}}{\hbar}\right) \tilde{\psi}_a(p') \quad \tilde{\psi}_a(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx' \exp\left(-\frac{ip'x^{;}}{\hbar}\right) \psi_a(x')$$

- The pair of equations is just what one expects from Fourier’s inversion theorem. The two representations are related with a Fourier transform!

THE MOMENTUM REPRESENTATION:

- We can solve in some cases the Schrödinger equation in the momentum representation. The following relations show us clearly how can we do it:

$$\hat{H}_{pos} = \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 + V(x) \quad \leftrightarrow \quad \hat{H}_{mom} = \frac{p^2}{2m} + V\left(i\hbar\frac{d}{dp}\right)^2$$

…in general the momentum representation formalism makes the solution of the Schroedinger equation complicated …