PHYS-453
3-THE UNCERTAINTY PRINCIPLE

…and other basic theorems of quantum mechanics

Average value of a physical quantity

The average value for an object's position is given by

$$\langle x \rangle = \int_{-\infty}^{+\infty} xP(x)dx = \int_{-\infty}^{+\infty} x\psi^{*}\psi\, dx = \int_{-\infty}^{+\infty} \psi^{*} x\psi\, dx = \int_{-\infty}^{+\infty} \psi^{*}\left(x\psi\right)dx$$

This expression shows us that the average value of any physical quantity A represented by an operator $\hat{A}$ is given by

$$\langle A \rangle = \int_{-\infty}^{+\infty} \psi^{*}\left(\hat{A}\psi\right)dx$$

The momentum operator

We can show that the operator which correspond to the physical quantity of the momentum is given by

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

Or generalizing in three dimensions of space:

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \qquad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \qquad \hat{p}_z = -i\hbar \frac{\partial}{\partial z}$$

$$\hat{\mathbf{p}} = -i\hbar \nabla$$

Uncertainty of a physical quantity

For any physical quantity A represented by an operator $\hat{A}$ the uncertainty is given by

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$$

with

$$\langle A \rangle = \int \psi^* \left(\hat{A}\psi \right) dx, \qquad \langle A^2 \rangle = \int \psi^* \left(\hat{A}^2 \psi \right) dx$$

Physical quantities depending on position and momentum

Any physical quantity can be written in terms of position and momentum $A(x,p)$. The average value of the quantity is given by

$$\left\langle A(x,p)\right\rangle = \int \Psi^* \hat{A}\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi dx$$

Heisenberg's Uncertainty Principle-a

- As we shall prove latter the uncertainty in the position and in the momentum satisfy the following relation known as *Heisenberg's Uncertainty Principle*:

$$\Delta x \cdot \Delta p \geq \hbar / 2$$

- This means that the more we know about a particle's position (small uncertainty) the less we know (large uncertainty) about momentum and vice versa.

Heisenberg's Uncertainty Principle-b

- The uncertainty principle is not an independent physical principle but a necessary consequence of the wave-particle duality and its statistical explanation.

Heisenberg's Uncertainty: A mathematical explanation-a

- Remember from mathematics that:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \quad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

- Also as we have shown in the class (for a real ψ:

$$\left(\Delta p\right)^2 = \hbar^2 \int \left|\psi'(x)\right|^2 dx$$

- The above relation says that the more "abrupt" (with large slopes) is a function the larger is the momentum uncertainty.

Heisenberg's Uncertainty: A mathematical explanation-b

- But a function with large slopes is "narrow" so it has a small position uncertainty. This qualitative discussion shows that the uncertainty in position "competes" with the uncertainty in momentum.
- Remember from (question 4, Handout 2) that:

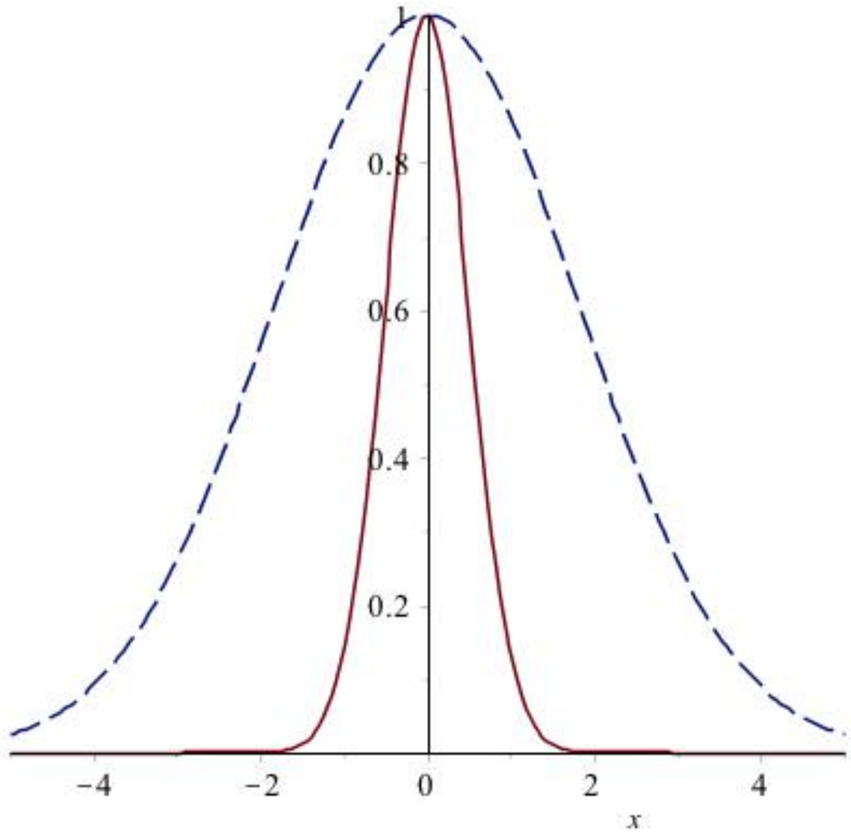
$$\psi(x) = \sqrt[4]{\frac{\lambda}{\pi}} e^{-\lambda x^2/2}, \quad \Delta x = \frac{1}{2\lambda}, \quad \Delta p = \hbar\sqrt{\frac{\lambda}{2}}$$

Heisenberg's Uncertainty: A mathematical explanation-b

- Remember from (question 4, Handout 2) that:

$$\psi(x) = \sqrt[4]{\frac{\lambda}{\pi}} e^{-\lambda x^2/2}, \quad \Delta x = \frac{1}{2\lambda}, \quad \Delta p = \hbar \sqrt{\frac{\lambda}{2}}$$

- In the plot you see the ψ for λ=4 (solid) and λ=0.3 (dashed).



Heisenberg's Uncertainty: A physical explanation-a

- If a particle has a fully determined momentum ($\Delta p=0$) then it is a wave with a definite wavelength $\lambda=h/p$. But a wave with a precise wavelength is a "plane wave" which has an infinite extend in space and thus $\Delta x=0$.
- From classical waves theory (Fourier analysis) we know that if we wish to create a "localized" wave (a "wavepacket") we must interfere a large number of sinusoidal waves with different wavelengths λ.

Heisenberg's Uncertainty: A physical explanation-c

- From the relation $p=\hbar k$ we get: Moreover we know that the more sinusoidal waves we interfere the more localized is the wave-packet. This happens because the waves interfere constructively in the "localization region" and destructively outside from this region.
- So Fourier analysis gives us the following result: $\Delta x \cdot \Delta k \approx 1$

Heisenberg's Uncertainty: A physical explanation-d

- From the relation $p = \hbar k$ we get:

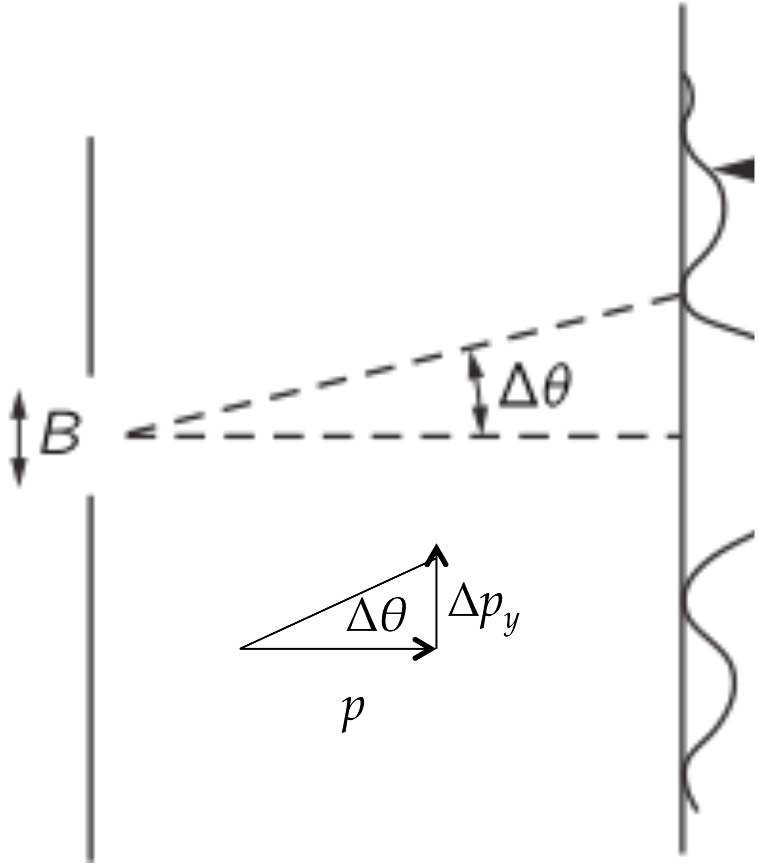
$$\Delta p = \hbar \cdot \Delta k \Rightarrow \Delta k = \Delta p / \hbar$$

$$\Delta x \cdot \Delta k \approx 1 \Rightarrow \Delta x \cdot \Delta p / \hbar \approx 1 \Rightarrow$$

$$\Delta x \cdot \Delta p \approx \hbar$$

A "gedanken" (thought) experiment

- When a particle passes through the slit it has a position along y-direction known to a precision $\Delta y \approx B$. The particles since they have wave properties they will suffer diffraction after passing through the slit. The beam will "open" by $\Delta\theta \approx \lambda / B$. Thus:



$$\Delta p_y = p \tan \Delta\theta \approx p\Delta\theta \approx p\lambda / B$$

$$= p(h/p)/B = h/B \underset{\Delta y \approx B}{\Rightarrow} \Delta y \cdot \Delta p_y \approx h$$

Theorem 1:

- Two quantum mechanical quantities A and B can be measured simultaneously with perfect precision only if their operators commute. That means only if $[A, B]=AB-BA=0$. On the contrary if $[A, B]\neq 0$, then the operators do not commute so the two quantities cannot be measured simultaneously with accuracy. Such quantities are called **complementary quantities** (like position and momentum).

The generalized uncertainty principle: Theorem 2:

- The product of the uncertainties of two complementary physical quantities can never be smaller than the half of the asolute average value of their commutator.

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\left\langle\left[A, B\right]\right\rangle\right|$$

- For $A=x$ and $B=p_x$ we know $[A, B]=[x, p_x]=i\hbar$, so

$$\Delta x \cdot \Delta p_x \geq \frac{1}{2}\left|\left\langle\left[x, p_x\right]\right\rangle\right| = \frac{1}{2}\left|\left\langle i\hbar\right\rangle\right| = \frac{1}{2}\left|i\hbar\right| = \frac{\hbar}{2}$$

- Similarly:

$$\Delta y \cdot \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \cdot \Delta p_z \geq \frac{\hbar}{2}$$

The time-energy uncertainty principle

- Time is not a dynamical quantity but rather a parameter both in classical and quantum mechanics. So there is no a time operator in quantum mechanics.
- The following statement holds: **The slower the variation of a system is (τ), the more precise the knowledge of its energy is, and vice versa.**

$$\Delta E \cdot \tau \geq \frac{\hbar}{2}$$

- We know in classical physics a similar relation for the frequency width and the time width of a pulse $\Delta\omega \cdot \Delta\tau \approx 1$.

The time-energy uncertainty principle

- To prove this uncertainty relation we need a rigorous definition of the characteristic evolution time τ of a quantity A. This is given as:

$$\frac{\Delta A}{\tau} = \left|\frac{d\langle A\rangle}{dt}\right| \Rightarrow \tau = \frac{\Delta A}{\left|d\langle A\rangle / dt\right|}$$

- This is actually the time we must wait in order the average value to has change by an amount equal to the standard deviation (or uncertainty) of A. To calculate the above quantity we need to know the quantity $d\langle A\rangle/\mathrm{d}t$.

The time-energy uncertainty principle

- This is given from the famous **Ehrenfest Theorem** which says that:

$$i\hbar \frac{d\langle \hat{A} \rangle}{dt} = \langle [\hat{A}, \hat{H}] \rangle$$

- The rate of change of the average value of a physical quantity is the average value of its commutator with the Hamiltonian.
- The Hamiltonian is the operator of the total energy

$$\hat{H} = \hat{p}^2 / 2m + \hat{V}$$