The finite square well-a

- The finite square well is the next problem that we are going to consider. The solution of Schroedinger equation is not that simple.
The finite square well-b

The potential in this problem has the form:

\[ V(x) = \begin{cases} 
0 & -a < x < a \\
V_0, & x < -a, \ x > a 
\end{cases} \]

Due to the symmetry of the potential the eigenfunctions will be alternatelly even and odd.

Hint: We are looking for bound states in the well. Thus, state for which

\[ 0 < E < V_0 \]
Solution of Schroedinger equation

We can show that the solutions of Schroedinger equation in each of the regions A, B and C are given by:

\[ A: \quad \psi_A'' + \left( \varepsilon - U_0 \right) \psi_A = \psi_A'' - \gamma^2 \psi_A = 0 \]

\[ B: \quad \psi_B'' + \varepsilon \psi_B = \psi_B'' + k^2 \psi_B = 0 \]

\[ C: \quad \psi_C'' + \left( \varepsilon - U_0 \right) \psi_C = \psi_C'' - \gamma^2 \psi_C = 0 \]

\[ \varepsilon = \frac{2mE}{\hbar^2}, \quad U_0 = \frac{2mV_0}{\hbar^2} \]

\[ \varepsilon = k^2, \quad U_0 - \varepsilon = \gamma^2 \quad \left( U_0 > \varepsilon \right) \]
**Even solutions-a**

In this case the solutions of Schroedinger equations are

\[ \psi_A = Ae^{\gamma x}, \quad \psi_B = B \cos kx, \quad \psi_C = Ae^{-\gamma x} \]

Where at region A and C we kept the relevant terms that go to zero at infinity. And we set the coefficients A and C equal in order to satisfy the even character of the solution

\[ \psi_A(x) = \psi_C(-x) \]
Even solutions-b

The coefficients A and B are calculated with the help of the following conditions:

\[
\psi_B(a) = \psi_C(a) \Rightarrow B \cos(ka) = Ae^{-\gamma a}
\]

\[
\psi_B'(a) = \psi_C'(a) \Rightarrow -Bk \sin(ka) = -\gamma Ae^{-\gamma a}
\]

\[
\tan(ka) = \frac{\gamma}{k}
\]
Odd solutions-a

In this case the solutions of Schroedinger equations are

\[ \psi_A = Ae^{\gamma x}, \quad \psi_B = B \sin kx, \quad \psi_C = -Ae^{-\gamma x} \]

Where at region A and C we kept the relevant terms that go to zero at infinity. And we set the coefficients \( C = -A \) in order to satisfy the odd character of the solution

\[ \psi_C(x) = -\psi_A(-x) \]
**Odd solutions-b**

The coefficients $A$ and $B$ are calculated with the help of the following conditions:

\[ \psi_B(a) = \psi_C(a) \Rightarrow B \sin(ka) = -Ae^{-\gamma a} \]
\[ \psi'_B(a) = \psi'_C(a) \Rightarrow Bk \cos(ka) = \gamma Ae^{-\gamma a} \]
\[ \tan(ka) = -\frac{k}{\gamma} \]

**Important note:** The coefficients $A$, $B$ and $C$ in the even solutions are different than the ones in the odd solutions! We kept the same symbols for simplicity.
Graphical solutions of the eigenvalues-a

The two equations which they will give us the energy spectrum are:

\[ \tan(ka) = \frac{\gamma}{k} \quad \tan(ka) = -\frac{k}{\gamma} \]

This can occur because \( k \) and \( \gamma \) depend on energy. But the analytic solution is impossible. We chose a graphical solution.
Graphical solutions of the eigenvalues

- We chose a parameter $\theta$ related to $k$ and $\gamma$ by:

\[ k = \sqrt{U_0} \cos \theta \quad \gamma = \sqrt{U_0} \sin \theta \]

Which satisfy the relation \[ k^2 + \gamma^2 = U_0 \]

Since $k$ and $\gamma$ are positive the angle $\theta$ is limited in the region $0 \leq \theta \leq \pi / 2$
Graphical solutions of the eigenvalues-c

- The unknown eigenvalue $\varepsilon$ is expressed as a function of the angle $\theta$ as follows

$$\varepsilon = U_0 \cos^2 \theta \quad \text{or} \quad E = V_0 \cos^2 \theta$$

- And our eigenvalues eqs., take the form

$$\tan(ka) = \frac{\gamma}{k} = \tan \theta \quad \tan(ka) = -\frac{k}{\gamma} = \tan \left(\theta - \frac{\pi}{2}\right)$$
Graphical solutions of the eigenvalues-d

Solving the above eqs. We get:

\[ \cos \theta = \frac{1}{\lambda} \theta + n \frac{\pi}{2\lambda}, \quad n = 0, 1, 2, 3, ..., \quad \text{and} \quad \lambda = a \sqrt{U_0} \]

where even (odd) values for \( n \) correspond to solutions for even (odd) eigenfunctions.
Graphical solutions of the eigenvalues-e

The number $N$ of bound states is finite. It is determined from the following relation:

$$N = \left\lfloor \frac{\lambda}{\pi / 2} \right\rfloor + 1$$
The eigenfunctions of the finite square well look like the corresponding ones of the infinite square well, but, there is an important difference: they are not zero at $x=-a$ and $x=a$ but they have exponential “tails” inside the forbidden region.
Discussion-b
We see for the first time that a particle can penetrate into a region which is classically forbidden.

In wave physics we have seen something similar: the electromagnetic waves could penetrate into a conductor where they suffer a damping.

The fact that the particle can penetrate into this forbidden region gives the wrong impression that in quantum mechanics, sometimes, we can violate the principle of energy conservation.
But there is a misconception here: In classical mechanics both kinetic and potential energy can be simultaneously measured and their sum gives always the total energy of the particle.

In quantum mechanics this is not anymore true: the total energy cannot be separated, at any position, as a sum of a kinetic and a potential term since we cannot measure simultaneously the position and momentum. Thus to say that, at a point \( x \), \( E < V(x) \) is meaningless.
This “paradox” could be also seen in a different way: In the classically forbidden region the wavefunction has an exponential term \( \exp(-\gamma x) \). This corresponds to a penetration practically at a length \( l=1/\gamma \) (known as penetration length) out of the well.

If we try to make a measurement and be sure that the particle is out of the well then \( \Delta x < l \). But in this case the kinetic energy will be:
We see that a measurement which finds the particle in the forbidden region disturbs the energy at least by the amount needed to kick it out of the well!

\[
\frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2ml^2} = \frac{\hbar^2 \gamma^2}{2m} = \frac{\hbar^2}{2m} (U_0 - \varepsilon) = V_0 - E
\]
We can see that at the classical limit (very large mass, or very small Planck’s constant) the penetration length tends to zero, as it is expected.

At the strong quantum limit (very small mass or large Planck’s constant) it becomes very large!

*The lighter particles shows strong quantum behaviour!*

\[ l = \gamma^{-1} = \sqrt{\frac{\hbar^2}{2m(V_0 - E)}} \]
Discussion-

- We can also observe that the deeper the well the smaller the penetration length and vice versa. (As it is expected when the well tends to the infinite depth)
- Also the penetration length depends on the energy. The higher the energy the higher the penetration.