PHYS 404
Lecture 8: Fourier Series

Fourier Series

Introduction

- In mathematics, a Fourier series decomposes periodic functions or periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis.

Fourier Series
Historical background

- The Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768–1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli.
- Fourier introduced the series for the purpose of solving the heat equation in a metal plate, publishing his initial results in his 1807 "*Mémoire sur la propagation de la chaleur dans les corps solides*" (Treatise on the propagation of heat in solid bodies), and publishing his Théorie analytique de la chaleur in 1822.
- Early ideas of decomposing a periodic function into the sum of simple oscillating functions date back to the 3rd century BC, when ancient astronomers proposed an empiric model of planetary motions, based on deferents and epicycles.

DEFINITION-a

- A Fourier series may be defined as an expansion of a function or a representation of a function in a series of cosines and sines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

- With the coefficients of the series to be given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \qquad n = 0, 1, 2, \ldots$$

- The above analysis is for a function with period 2π.

DEFINITION-b

- If the function has a period *l*, then we make the following substitution $z = 2\pi x / l$

and we can have the following expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi x}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(n\frac{2\pi x}{l}\right)dx, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\left(n\frac{2\pi x}{l}\right)dx \qquad n = 0, 1, 2,...$$

preconditions

- The expansion in a Fourier series is valid provided that the function is *a piecewise regular function.*
- This means that the function has a finite number of discontinuities and a finite number of extreme values, maxima and minima
- These conditions (known as *Dirichlet conditions*) are *sufficient* but not necessary.
- The above expressions shows us that $f(x)$ is part of an infinite-dimensional Hilbert space with the orthogonal cosnx and sinnx as the basis.

The Dirichlet Conditions

Suppose that

1. $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

2. $f(x)$ is periodic outside $(-L, L)$ with period $2L$

3. $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series with Fourier coefficients converges to

a) $f(x)$ if x is a point of continuity

b) $f(x_0) = (1/2)\left[f(x_0+) + f(x_0-)\right]$ if x_0 is a point of discontinuity

Even and Odd functions-a

- In the Fourier corresponding to an odd function, only sine terms can be present.
- In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant which we shall consider a cosine term) can be present.
- Usually the above are described by the term *half range Fourier sine and cosine series*.

Half range Fourier cosine series

Let an even function $F(x)$ in the range from -π to π. The function can be expanded in a half range cosine series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \qquad n = 0, 1, 2, \ldots$$

Half range Fourier sine series

Let an odd function $G(x)$ in the range from $-\pi$ to π. The function can be expanded in a half range sine series as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx, \qquad n = 1, 2, \ldots$$

Parseval's Identity

- If a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions, then:

$$\frac{1}{L}\int_{-L}^{L}\left\{f(x)\right\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Different forms of Fourier Series

- We may show if we express the cosines and sines in exponential form that a Fourier series me be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2}\left(a_n - ib_n\right), \quad c_{-n} = \frac{1}{2}\left(a_n + ib_n\right) \quad \left(n > 0\right)$$

$$c_0 = a_0 / 2$$

Prove that: $c_n = \frac{1}{2\pi}\int_0^{2\pi} f\left(x\right)e^{-inx}dx$

to be noted …

When dealing with Fourier expansions we must have in mind the following relations.

$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0 \\ 0, & m = 0 \end{cases}$$

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0 \\ 2\pi, & m = n = 0 \end{cases}$$

$$\int_0^{2\pi} \sin(mx)\cos(nx)dx = 0, \qquad \forall\ m,\ n$$

$$\int_0^{2\pi} \left(e^{imx}\right)^* e^{inx}dx = 2\pi\delta_{m,n}$$

to be noted …

- Note carefully that any interval of the form

$$x_0 \leq x \leq x_0 + 2\pi$$

will be equally satisfactory. Frequently we use the interval .

- For the exponential form the orthogonality is expressed as $-\pi \leq x \leq \pi$.

$$\int_0^{2\pi} \left(e^{imx}\right)^* e^{inx} dx = 2\pi\delta_{m,n}$$

Power content of a periodic function

- The power content of a periodic function $f(t)$ with period T is defined as

$$\frac{1}{T}\int_{-T/2}^{T/2}\left[f\left(t\right)\right]^2 dt$$

If we assume that the function $f(t)$ is a voltage across a resistor then from the above relation we could get the average power delivered to the resistor. It can be shown that,

$$\frac{1}{T}\int_{-T/2}^{T/2}\left[f\left(t\right)\right]^2 dt = \sum_{n=-\infty}^{\infty}\left|c_n\right|^2$$

Advantages of Fourier Series-a

- One of the advantages of a Fourier representation over some other representation, such as Taylor series, is that it may represent a discontinuous function. An example is the sawtooth wave.
- Another advantage is the representation of periodic functions.

Advantages of Fourier Series-b

- Another advantage is the solution of the problem of the dynamics of an oscillating particle subject to a periodic driving force. The Fourier expansion of the driving force then gives us the fundamental term and a series of harmonics. The (linear) DE may be solved for each of these harmonics individually which is far easier than dealing with the original driving force. Then, as long as the DE is linear all the solutions may be added together to obtain the final solution.

Advantages of Fourier Series-d

If a function $f(x)$ is not periodic it evidently cannot be expanded in a Fourier series for *all* values of x. Nevertheless we can find a Fourier series to represent it over any range of width 2π, say from $-\pi$ to π or from 0 to 2π.

For consider a new function, say $g(x)$, obtained by taking the values of $f(x)$ in the given range and repeating them outside the range at intervals of 2π. The function $g(x)$ is, by construction, periodic with a period 2π, and it can therefore be expanded in a Fourier series for all values of x. Since $g(x)= f(x)$ in the given range, it follows that the sum of this series is equal to $f(x)$ at all points in the given range, though not of course at points outside the range.