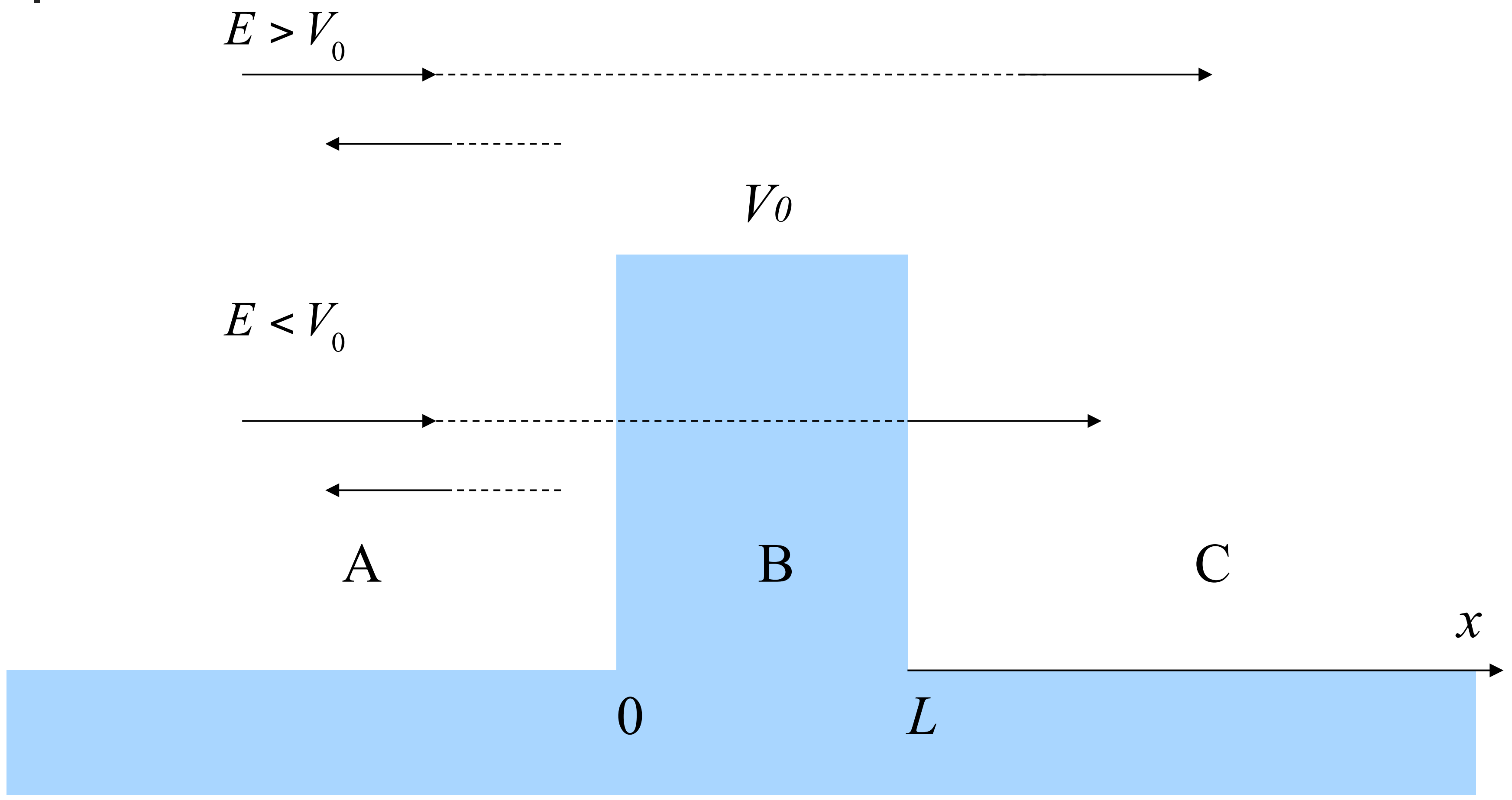
PHYS-454

The orthogonal potential barrier & the tunneling effect



$E > V_0$
V_0
$E < V_0$
A
B
C
x
0
L

The orthogonal potential barrier-a

- **A) The case where** $E > V_0$

The Schroedinger equation takes the form in the two regions A and B:

$$A: \qquad \psi''_A + \varepsilon\psi_A = \psi''_A + k^2\psi_A = 0$$

$$B: \qquad \psi''_B + \left(\varepsilon - U_0\right)\psi_B = \psi''_B + k'^2\psi_B = 0$$

$$C: \qquad \psi''_C + \varepsilon\psi_C = \psi''_C + k^2\psi_C = 0$$

$$\varepsilon = \frac{2mE}{\hbar^2}, \qquad U_0 = \frac{2mV_0}{\hbar^2}$$

$$k^2 = \varepsilon, \qquad k'^2 = \varepsilon - U_0 > 0$$

The orthogonal potential barrier-b

In this case the general solutions of Schroedinger equations are:

$$\psi_A = e^{ikx} + Ae^{-ikx}$$

$$\psi_B = B_+ e^{ik'x} + B_- e^{-ik'x}$$

$$\psi_C = Ce^{ikx}$$

The orthogonal potential barrier-c

By solving the above equations we get

$$A = \frac{i\left(k'^2 - k^2\right)\sin\left(k'L\right)}{2k'k\cos\left(k'L\right) - i\left(k'^2 + k^2\right)\sin\left(k'L\right)}$$

$$R = \left|A\right|^2 = \frac{U_0^2 \sin^2\left(k'L\right)}{U_0^2 \sin^2\left(k'L\right) + 4k'^2k^2}$$

$$T = 1 - R = \frac{4k'^2k^2}{U_0^2 \sin^2\left(k'L\right) + 4k'^2k^2} = \frac{4\varepsilon\left(\varepsilon - U_0\right)}{U_0^2 \sin^2\left(L\sqrt{\varepsilon - U_0}\right) + 4\varepsilon\left(\varepsilon - U_0\right)}$$

$$= \frac{4E\left(E - V_0\right)}{V_0^2 \sin^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{E - V_0}\right) + 4E\left(E - V_0\right)}$$

The orthogonal potential barrier-d

- **B) The case where** $E < V_0$

$$R = \frac{U_0^2 \sinh^2(\gamma L)}{U_0^2 \sinh^2(\gamma L) + 4\gamma^2 k^2}$$

$$\gamma = \sqrt{U_0 - \varepsilon}$$

$$= \frac{U_0^2 \sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{U_0 - \varepsilon}\right)}{U_0^2 \sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{U_0 - \varepsilon}\right) + 4\left(U_0 - \varepsilon\right)\varepsilon}$$

$$\sinh \gamma L = \frac{1}{2}\left(e^{\gamma L} - e^{-\gamma L}\right)$$

$$= \frac{V_0^2 \sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{V_0 - E}\right)}{V_0^2 \sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{V_0 - E}\right) + 4\left(V_0 - E\right)E}$$

The orthogonal potential barrier-e

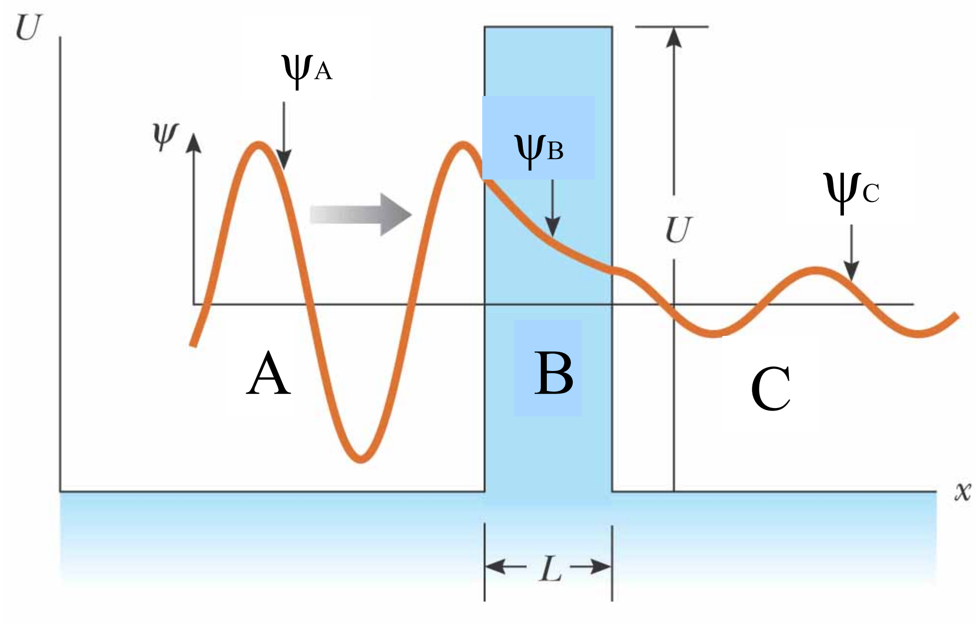
$$T = 1 - R = \frac{4\gamma^2 k^2}{U_0^2 \sinh^2(\gamma L) + 4\gamma^2 k^2}$$

$$= \frac{4(U_0 - \varepsilon)\varepsilon}{U_0^2 \sinh^2\left(L\sqrt{U_0 - \varepsilon}\right) + 4(U_0 - \varepsilon)\varepsilon}$$

$$= \frac{4(V_0 - E)E}{V_0^2 \sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{V_0 - E}\right) + 4(V_0 - E)E}$$

Discussion-a
The tunneling effect

- The most exciting aspect of the solution is the possibility of a particle to traverse a classicaly forbiden region: this is the *tunneling effect*



Discussion-b
Tunneling effect and energy

- It can be shown that the transmission coefficient *T*, in the case of a wide and high potential barrier, can get the form

$$T(E) \approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)\exp\left(-2\gamma L\right)$$

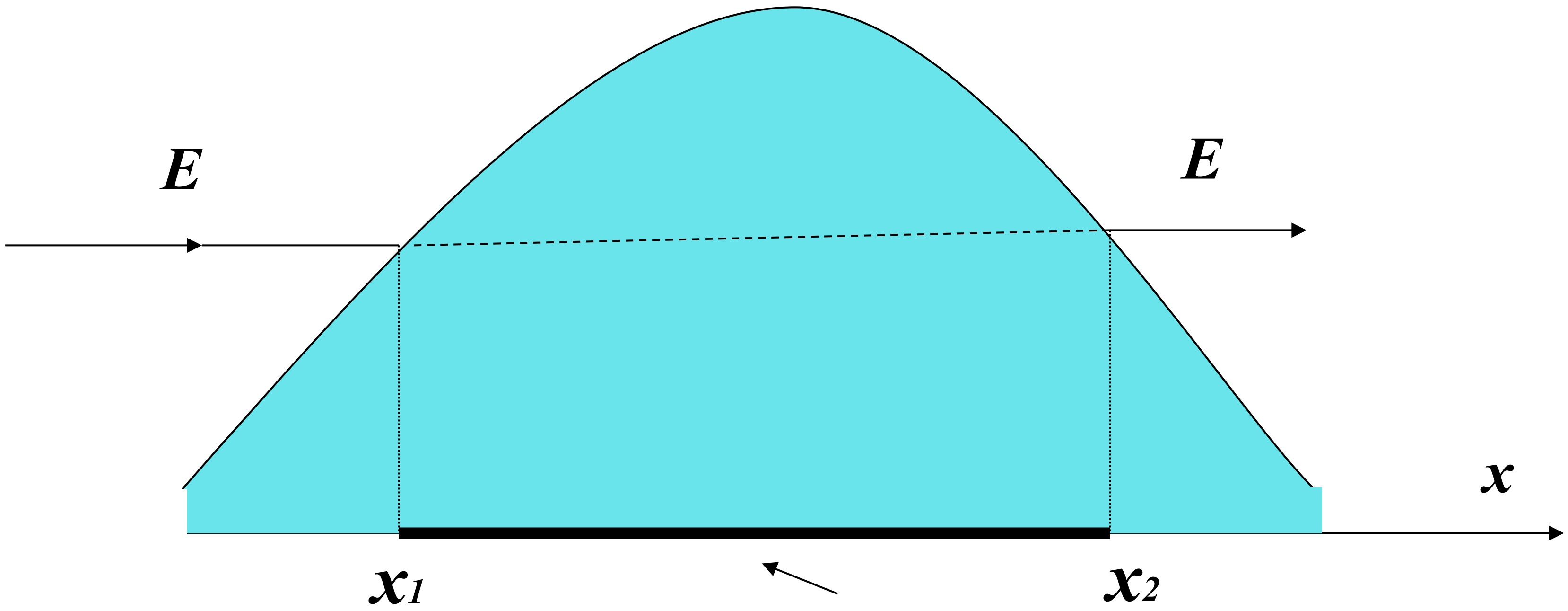
$$= 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)\exp\left(-2L\sqrt{2m\left(V_0 - E\right)/\hbar^2}\right)$$

Discussion-c
Tunneling effect and energy

- This shows an exponential dependence on the energy changes of the particle. For small energy changes (like those occurring in alpha decay) the factor in front of the exponential can be considered constant, compared to the exponential term, and almost equal to 1. In this case the transmission coefficient gets the form

$$T(E) \approx \exp\left(-2\gamma L\right), \qquad \gamma = \sqrt{2m\left(V_0 - E\right)/\hbar^2}$$

Discussion-d
Tunneling effect - generic potential barrier



Classicaly forbiden region

Discussion-e
Tunneling effect - generic potential barrier

- Within the classical forbidden region the damping coefficient γ is not anymore constant but it will depend on position, i.e.

$$\gamma\left(x\right)=\sqrt{U(x)-\varepsilon}=\sqrt{2m\left(V_0-E\right)/\hbar^2}$$

- This forces us to consider an average value for this coefficient

Discussion-f
Tunneling effect - generic potential barrier

$$\bar{\gamma} = \frac{1}{L}\int_{x_1}^{x_2} \gamma(x)\,dx \qquad (L = x_2 - x_1)$$

$$T \approx \exp\left(-2\bar{\gamma}L\right) = \exp\left(-2\int_{x_1}^{x_2} \gamma(x)\,dx\right)$$

$$= e\,\mathrm{xp}\left(-2\int_{x_1}^{x_2} \sqrt{\frac{2m\left(V(x) - E\right)}{\hbar^2}}\,dx\right)$$

Gamow approximation

x_1, x_2 are points where: $E = V(x_1) = V(x_2)$

Discussion-g
Tunneling effect - the role of mass

- From the above formulae it is obvious that tunneling effect is very sensitive to the mass of the particle. The lighter a particle the higher its transmission coefficient.

Discussion-h
Tunneling effect - resonances

- If we see the formula for the transmission coefficient in the case of $E > V_0$

$$T = 1 - R = \frac{4k'^2k^2}{U_0^2 \sin^2\left(k'L\right) + 4k'^2k^2}$$

- We can easily see that we have full transmission (T=1) when

$$\sin\left(k'L\right) = 0 \Rightarrow k'L = n\pi$$

Discussion-i
Tunneling effect - resonances

- Given that $k'^2 = \varepsilon - U_0 = 2m\left(E - V_0\right)/\hbar^2$

we can easily prove that the *resonance energies* are given by

$$E_n = V_0 + \frac{\hbar^2\pi^2}{2mL^2}n^2$$

This means that **we have full transmission every time the energy of the particle has an energy higher than that of the barrier at amounts equal to the eigenvalues of the infinite well!**

Discussion-j
Tunneling effect - resonances

- Given, also that $k = 2\pi / \lambda'$ then we get

we can easily prove that the *resonance energies* are given by

$$k'L = n\pi \Rightarrow L = n\frac{\lambda'}{2}$$

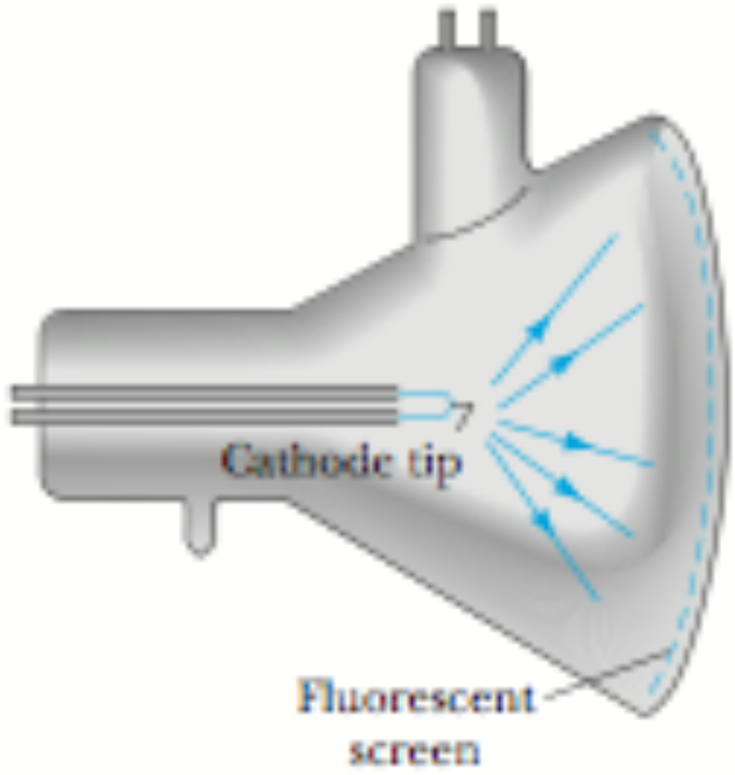
This means that **we have resonance every time that the wavelength of the particle is such that the width of the barrier can accommodate integer half-wavelengths of the particle!** This is equivalent to the resonance condition when a ligth passes through a thin film.

Discussion-k
Tunneling effect - the e/m wave analogue

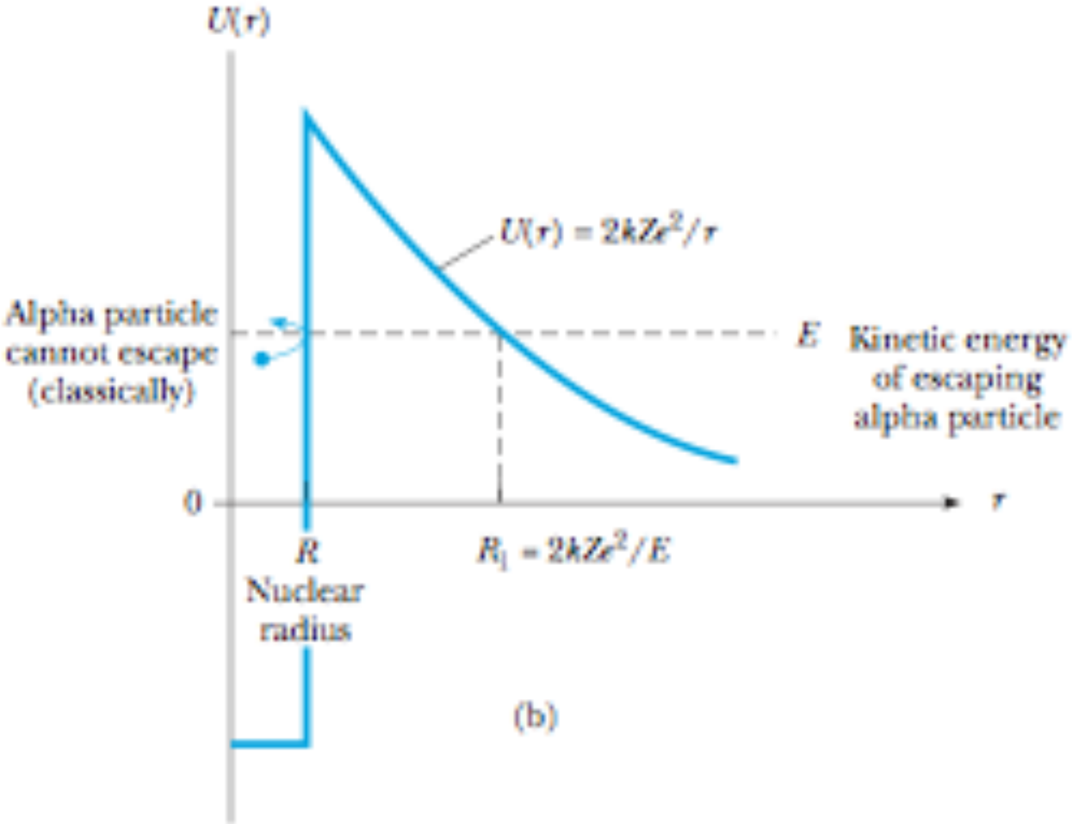
- (a) The e/m wave analogue for the case of an infinite potential barrier (**total internal reflection**)
- (b) The e/m wave analogue for the case of tunneling through a finite potential barrier (**frustrated total internal reflection**)

Applications-a
Field emission microscope

- The intense electric field at the cathode tip allows electrons to tunnel through the work function barrier at the surface. Since the tunneling probability is sensitive to the exact details of the surface, the number of escaping electrons varies from point to point, thus providing a picture of the surface under study.



Applications - b
The mechanism of alpha decay



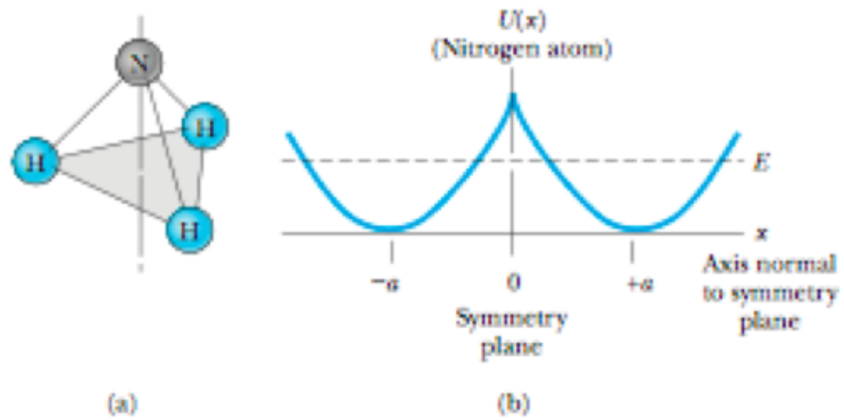
Applications - c
The mechanism of alpha decay

- Tunneling effect helped us to clarify the two most puzzling aspects of α-decay:
- a) all α-particles emitted from any source emerge with kinetic energies in the same narrow range, from 4 to 9 MeV
- b) the half-life of the mitters varries in an enormous range - more than 20 orders of magnitude! - according to the emitting element.
- The transmission coefficient in α-decay is given by:

$$T(E) = \exp\left\{-4\pi Z\sqrt{\frac{E_0}{E}} + 8\sqrt{\frac{ZR}{r_0}}\right\}$$

$$r_0 = \hbar^2 / m_a k e^2 = 7.25 \text{ fm}, \qquad E_0 = ke^2 / 2r_0 = 0.0993 \text{ MeV}$$

Applications-d
The mechanism of alpha decay



- The ammonia molecule with the nitrogen at the apex of a pyramid whose base is the equilateral triangle formed by three hydrogen atoms. There are two symmetric equivalent positions for the nitrogen atom which can tunnel back and forth and oscillate between them.

Applications - e
The black holes

- We know that nothing can escape the gravity of a black hole. In 1974 Stephen Hawking showed that black holes emitts a variety of particles by a mechanism involving tunneling through the gravitational potential barrier surrounding the black hole.