MATH203 Calculus

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Independent of path

Theorem 1

If $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuous on an open connected region D, then the line integral $\int_C \mathbf{F} \cdot dr$ is independent of path if and only if $\mathbf{F}$ is conservative that is $\mathbf{F}(x, y) = \nabla f(x, y)$ for some scalar function.

Theorem 2: Fundamental theorem of line integrals

Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be continuous on an open connected region D, and let C be a piecewise smooth curve in D with endpoint $A(x_1, y_1)$ and $B(x_2, y_2)$. If $\mathbf{F}(x, y) = \nabla f(x, y)$, then

$$\int_C M(x, y)dx + N(x, y)dy = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F} \cdot dr = \Big[f(x, y)\Big]_{(x_1,y_1)}^{(x_2,y_2)}$$

Independent of path

Theorem 3

If $M(x,y)$ and $N(x,y)$ have continuous first partial derivatives on a simply connected region D, then the line integral $\int_C M(x,y)dx + N(x,y)dy$ is independent of path in D if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 1: Let $\mathbf{F}(x,y) = (2x + y^3)\mathbf{i} + (3xy^2 + 4)\mathbf{j}$

(a) Show that $\int_C \mathbf{F} \cdot dr$ is independent of path.

(b) $\int_{(0,1)}^{(2,3)} \mathbf{F} \cdot dr$.

Independent of path

Example 2: Show that $\int_C (e^{3y} + y^2 \sin x)dx + (3xe^{3y} - 2y \cos x)dy$ is independent of path in a simply connected region.

Example 3: Determine whether $\int_C x^2 y dx + 3xy^2 dy$ is independent of path.

Example 4: Let $\mathbf{F}(x, y, z) = y^2 \cos x \mathbf{i} + (2y \sin x + e^{2z})\mathbf{j} + 2ye^{2z}\mathbf{k}$

(a) Show that $\int_C \mathbf{F} \cdot dr$ is independent of path, and find a potential function f of $\mathbf{F}$.

(b) If $\mathbf{F}$ is a force field, find work done by $\mathbf{F}$ along any curve C from $(0, 1, \frac{1}{2})$ to $(\frac{\pi}{2}, 3, 2)$.

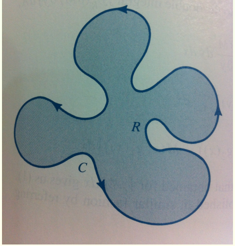
Green's Theorem

Green's theorem

Let C be a piecewise-smooth simple closed curve, and let R be the region consisting of C and its interior. If M and N are continuous functions that have continuous first partial derivatives throughout an open region D containing R, then

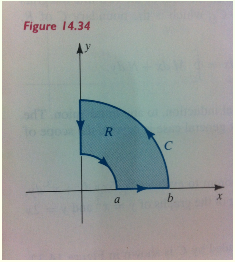
$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Figure 14.28



Green's Theorem

Note: Note the line integral is independent of path and hence is zero $\int_C \mathbf{F} \cdot dr = 0$ for every simple closed curve C.
Examples: (1) Use Green's theorem to evaluate $\oint_C 5xydx + x^3dy$, where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points $(0, 0)$ and $(2, 4)$.
(2) Use Green's theorem to evaluate $\oint_C 2xydx + (x^2 + y^2)dy$, if C is the ellipse $4x^2 + 9y^2 = 36$.
(3) Evaluate $\oint_C (4 + e^{\cos x})dx + (\sin y + 3x^2)dy$, if C the boundary of the region R between quarter-circles of radius a and b and segment on the $x-$ and $y-$axes, as shown in Figure.



Green's Theorem

Theorem

If a region R in the $xy-$plane is bounded by a piecewise-smooth simple closed curve C, then the area A of R is

$$\iint_R dA = \oint_C x\,dy \qquad \text{(i)}$$
$$= -\oint_C y\,dx \qquad \text{(ii)}$$
$$= \frac{1}{2}\oint_C x\,dy - y\,dx. \qquad \text{(iii)}$$

Examples: (1) Find the area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

(2) Find the area of the region bounded by the graphs of $y = 4x^2$ and $y = 16x$.

Examples: (1) Show that $\int_C \mathbf{F} \cdot dr$ is independent of paths by finding a potential function f
(a) $\mathbf{F}(x,y) = (3x^2y+2)\mathbf{i} + (x^3+4y^3)\mathbf{j}$
(b) $\mathbf{F}(x,y) = (2xe^{2y}+4y^3)\mathbf{i} + (2x^2e^{2y}+12xy^2)\mathbf{j}$

(2) Show that $\int_C \mathbf{F} \cdot dr$ is independent of paths and find its value
(a) $\int_{(-1,2)}^{(3,1)} (y^2+2xy)dx + (x^2+2xy)dy$
(b) $\int_{(4,0,3)}^{(-1,1,2)} (yz+1)dx + (xz+1)dy + (xy+1)dz$

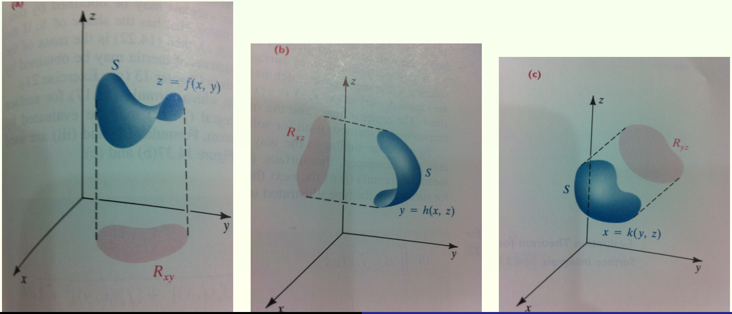
(3) Use Green's theorem to evaluate the line integral
(a) $\oint_C x^2y^2dx + (x^2-y^2)dy$, where C is the square with vertices $(0,0),(1,0),(1,1),(0,1)$.
(b) $\oint_C xydx + (x+y)dy$, where C is the circle $x^2+y^2=1$.
(c) $\oint_C xydx + \sin y dy$, where C is the triangle with vertices $(1,1),(2,2),(3,0)$.

Surface Integrals

Surface Integrals

Line integrals are evaluated along curves, Double and triple integral are defined on regions in two and three dimensions, respectively. In this topic we consider integrals of function over surfaces.

$$\iint_S g(x,y,z)ds = \lim_{\|P\|\to 0} \sum_k g(x_k, y_k, z_k)\Delta T_k.$$



Surface Integrals

Evaluation Theorem for Surface integrals

(i) $$\iint\limits_{S} g(x,y,z)dS = \iint\limits_{R_{xy}} g(x,y,f(x,y))\sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}dA$$

(ii) $$\iint\limits_{S} g(x,y,z)dS = \iint\limits_{R_{xz}} g(x,h(x,y),z)\sqrt{[h_x(x,y)]^2+[h_z(x,y)]^2+1}dA$$

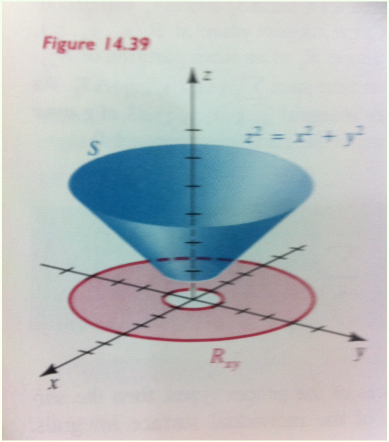
(iii) $$\iint\limits_{S} g(x,y,z)dS = \iint\limits_{R_{yz}} g(k(x,y),y,z)\sqrt{[k_y(x,y)]^2+[k_z(x,y)]^2+1}dA$$

Examples: (1) Find the area of the ellipse $(x^2/a^2)+(y^2/b^2)=1$.

(2) Find the area of the region bounded by the graphs of $y=4x^2$ and $y=16x$.

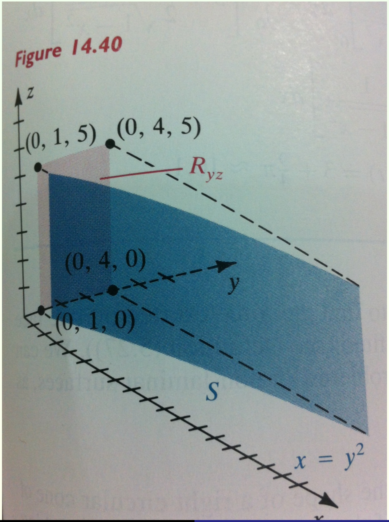
Surface Integrals

Examples: (1) Evaluate $\iint_S x^2 z dS$ if S is the portion of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 4$.



Surface Integrals

(2) Evaluate $\iint_S (xz/y)dS$ if S is the portion of the cylinder $x = y^2$ that lies in the first octant between the planes $z = 0, z = 5, y = 1$, and $y = 4$.



Surface Integrals

(3) Evaluate $\iint_S (z+y)dS$ if S is the part of the graph of $z=\sqrt{1-x^2}$ in the first octant between the $xz-$plane and the plane $y=3$.