Lecture 1

In this course, we will learn methods of geometry in modern physics. Historically, mathematics and physics influenced each other in their developments. We know, for example, that Newton invented calculus and classical mechanics simultaneously. However, in the mid 20th century, there was a period when interactions between physicists, in particular high energy physicists, and mathematicians almost stopped. There were two main reasons. One was on the mathematics side. The Bourbaki movement.

Another was on the physics side. Quantum field theory was conceived by Heisenberg and Pauli in 1929 and Feynman, Schwinger, and Tomonaga established the renormalization procedure to make sense of QED. However, there was not mathematical formulation of quantum field theory. In fact, there is still no rigorous mathematical foundation for it, except for special cases, some of which we will see later in this course. Making sense of quantum Yang-Mills theory is posed as one of the seven Millennium problems by the Clay Mathematics Institute.

The situtation has dramatically changed since the 1980's. It is because of the use of supersymmetry. Supersymmetry brought mathematics and physics closer since it is close to the language mathematicians use to describe geoemtric concepts.

Why supersymmetry?

Differential forms play important roles. Differential forms are composed of objects which do not commute with each other but anti-commute, just like fermions. In fact, as we will learn in the next couple of lectures, at each point on a manifold, the space of differential forms can be regarded as the Fock space of fermons. On the other hand, coordinates of a manifold commute with each other and behave like bosons. Combining the fermions and the bosons together, the total space of differential forms will become a Hilbert space of a supersymmetric system.

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In this first lecture, we will study **exterior product algebra**. This is going to be a space of differential operators at a given point on a manifold.

Consider an n-dimensional vector space V with basis vectors $\{e_i\}$ $(i = 1, ..., n)$. Each element v of V can be expanded as

$$v = \sum_{i=1}^{n} v^i e_i = v^i e_i,$$

where I used the *Einstein convension*: a repeated index should be summed over. The coefficient v^i are called components of the vector v.

For a given vector space V, we can consider its dual space V^*. It is a space of linear functions on V,

$$\omega \in V^* : \qquad V \to \mathbf{R}, \quad \text{(linear)}.$$

The dimensions of V and V^* are the same.

For a given set of basis vectors $\{e_i\}$, we can define the dual basis $\{e^i\}$ by

$$e^i : \quad e_j \to e^i(e_j) = \delta^i_j.$$

Any element ω of V^* can be expanded as

$$\omega = \omega_i e^i,$$

so that

$$\omega(v) = \omega_i v^i.$$

Question 1: Show $V^{**} = V$.

We can consider a tensor product of V^* to define a space of multi-linear functions, $V^* \otimes V^* \otimes \cdots \otimes V^*$. We can expand its element as

$$\omega = \omega_{i_1 i_2 \cdots i_k} e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_k}.$$

We can also consider mixed tensors in $\otimes^k V^* \otimes^l V$.

forms

Let $\mathcal{S}_k$ be the symmetric group of k object,

$$\sigma \in \mathcal{S}_k : \ (1, 2, ..., k) \to (\sigma(1), \sigma(2), ..., \sigma(k)).$$

We call $\omega \in \otimes^k V^*$ as a k-form iff (this should read as "if and only if"),

$$\omega(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}) = (-1)^\sigma \omega(v_1, v_2, ..., v_k).$$

Here $(-1)^\sigma$ is equal to $+1$ or -1 depending on whether σ can be expressed as a product of an even or odd number of permutations (exchange of two elements).

wedge product

We can take a product of a k-form and an l-form to make a $(k+l)$-form. Let us start with a product of 1-forms. If α^a's are 1-forms (i.e., $\alpha^a \in V^*$), their wedge product are defined by

$$\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k(v_1, v_2, ..., v_k) = \det(\alpha^a(v_b)).$$

More generally, if α is a k-form and β is an l-form,

$$(\alpha \wedge \beta)(v_1, ..., v_{k+l}) = \frac{1}{k!l!} \epsilon^{i_1 \cdots i_{k+l}} \alpha(v_{i_1}, ..., v_{i_k}) \beta(v_{i_{k+1}}, ..., v_{i_{k+l}}).$$

We can choose a basis of the space of k-forms, $\wedge^k V^*$, as

$$e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

Any k-form α can be expanded as

$$\alpha = \frac{1}{k!} \alpha_{i_1, ..., i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

Question 2: When $\alpha \in V^*$ and $\beta \in V^* \wedge V^*$, express $\alpha \wedge \beta$ in components.

Question 3: When $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$, show

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

There are transformations that map a k-form to a $(k+1)$-form and a $(k-1)$-form. The former is easy,

$$\alpha \in \wedge^k V^* \to e^i \wedge \alpha \in \wedge^{k+1} V^*.$$

The latter is defined as follows.

interior product

For $\alpha \in \wedge^k V^*$ and $u \in V$, we define $i(u)\omega \in \wedge^{k-1} V^*$ as,

$$(i(u)\alpha)(v_1, ..., v_{k-1}) = \alpha(u, v_1, ..., v_{k-1}).$$

Question 4-1: Show $i(v_1)i(v_2) = -i(v_2)i(v_1)$.

Question 4-2: When α is a k-form and β is an l-form, show

$$i(v)(\alpha \wedge \beta) = (i(v)\alpha) \wedge \beta + (-1)^k \alpha \wedge (i(v)\beta).$$

metric

We can do more if there is a metric $g \in V^* \otimes V^*$, which is symmetric, $g(v_1, v_2) = g(v_2, v_2)$ and non-degenerate, namely $g(v, u) = 0$ for any v implies $u = 0$. The metric g can be expanded as

$$g = g_{ij} e^i \otimes e^j, \qquad g_{ij} = g(e_i, e_j).$$

When a metric g is given, we can identify V with V^*, namely there is a map $\tilde{g} : V \to V^*$,

$$\tilde{g}(v) : u \to g(v, u).$$

Question 5-1: Show $\tilde{g}(e_i) = g_{ij} e^j$.

Question 5-2: What is the inverse of the map $\tilde{g}$?

Since g_{ij} is symmetric, we can always diagonalize it. In fact, we can choose a basis $\{\mathcal{O}^i\}$, so that

$$g(\mathcal{O}i, \mathcal{O}j) = \pm\delta_{ij}.$$

In this basis,

$$g = \mathcal{O}^1 \otimes \mathcal{O}^1 + \cdots + \mathcal{O}^k \otimes \mathcal{O}^k - \mathcal{O}^{k+1} \otimes \mathcal{O}^{k+1} - \cdots - \mathcal{O}^n \otimes \mathcal{O}^n.$$

In this case, we call that g is of signature $(k, n-k)$.

volume element

is defined by

$$\eta = \mathcal{O}^1 \wedge \mathcal{O}^2 \wedge \cdots \wedge \mathcal{O}^k.$$

Question 6: Show that, for a generic basis $\{e_i\}$,

$$\eta = \pm\sqrt{|\det g(e_i, e_j)|}\, e^1 \wedge \cdots \wedge e^k,$$

where the sign is determined by the relative orientation of $\{\mathcal{O}_i\}$ and $\{e_i\}$.

Hodge $*$ operator

The Hodge $*$ operator is a map $\wedge^k V^* \to \wedge^{n-k} V^*$ defined by

$$*\alpha(v_{k+1}, \cdots, v_n)\eta = \alpha \wedge \tilde{g}(v_{k+1}) \wedge \cdots \tilde{g}(v_n).$$

Show that, in components, the Hodge $*$ operator can be expressed as

$$(*\alpha)_{i_{k+1}\cdots i_n} = \pm \frac{1}{k!} \alpha_{j_1 \cdots j_k} \eta^{j_1 \cdots j_n} g_{i_{k+1} j_{k+1}} \cdots g_{i_n j_n},$$

where the sign in the right-hand side is determined by the signature of the metric g. When the mertic is positive definite,

$$* * \omega = (-1)^{k(n-k)} \omega.$$

fermions

Consider free fermions obeying the anti-commutation relations,

$$\{\psi_i, \psi_j\} = 0, \quad \{\bar{\psi}^i, \bar{\psi}^j\} = 0, \quad \{\psi_i, \bar{\psi}^j\} = \delta_i^j \qquad (i, j = 1, ..., n).$$

The Fock space is built, starting with the vacuum state $|0\rangle$ annihilated by all ψ_i.

We can identify the Fock space with the space of forms $\sum_{k=0}^n \wedge^k V^*$, with the identification,

$$|0\rangle \leftrightarrow 1, \quad \bar{\psi}^i \leftrightarrow e^i \wedge, \quad \psi_i \leftrightarrow i(e_i).$$

When we have a metric g on V, we can use it to define an inner product in the Fock space. It is defined so that

$$(\psi_i)^\dagger = g_{ij} \bar{\psi}^j,$$

and

$$(\bar{\psi}^i|0\rangle, \bar{\psi}^j|0\rangle) = \langle 0|(\bar{\psi}^i)^\dagger \bar{\psi}^j|0\rangle = g^{ik} \langle 0|\psi_k \bar{\psi}^j|0\rangle = g^{ij}.$$