

# MATH203 Calculus

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# Outline

- Definition of sequences.
- Definition of convergent sequence.
- Definition of divergent sequence.
- Definition of constant sequence.
- Theorem 1.
- L' Hopital's rule.
- Theorem 2 (Properties of limits of sequences).
- Theorem 3 (Absolute value).

## Definition of sequences

A sequence is a function whose domain is the set of positive integers. It is denoted by  $\{a_n\} = a_1, a_2, a_3, \dots, a_n, \dots$  (entire seq) and  $\{a_n\} = a_1, a_2, a_3, \dots, a_n$  (finite seq).

**Example:** Find the first four terms and  $n$ th term of each:

(a)  $\left\{\frac{n}{n+1}\right\}$       (b)  $\{2 + (0.1)^n\}$       (c)  $\{(-1)^{n+1} \frac{n^2}{3n-1}\}$

(d)  $\{4\}$       (e)  $a_1 = 3$  and  $a_{k+1} = 2a_k$  for  $k \geq 1$ .

### Definition of convergent sequence (c'gt)

A sequence  $\{a_n\}$  has a limit  $L$ , or converges to  $L$  denoted by either  $\lim_{n \rightarrow \infty} a_n = L$  or  $a_n \rightarrow L$  as  $n \rightarrow \infty$ .

### Definition of divergent sequence (d'gt)

A sequence  $\{a_n\}$  is called if

- $\lim_{n \rightarrow \infty} a_n$  does not exist.
- $\lim_{n \rightarrow \infty} a_n = +\infty$  or  $\lim_{n \rightarrow \infty} a_n = -\infty$ .

### Definition of constant sequence

A  $\{a_n\}$  is constant if  $a_n = c$  for every  $n$ ,  $c \in \mathbb{R}$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c = c$ .

### Theorem 1

Let  $\{a_n\}$  be a sequence and  $f$  be a function such that

- $f(n) = a_n$
- $f(x)$  exists for every real number  $x \geq 1$

then

- ① If  $\lim_{x \rightarrow \infty} f(x) = L$ , then  $\lim_{n \rightarrow \infty} f(n) = L$
- ② If  $\lim_{x \rightarrow \infty} f(x) = \infty$  (or  $-\infty$ ), then  $\lim_{n \rightarrow \infty} f(n) = \infty$  (or  $-\infty$ ).

### Examples:

(1) If  $a_n = 1 + \left(\frac{1}{n}\right)$ , determine whether  $\{a_n\}$  converges or diverges.

(2) Determine whether  $\{a_n\}$  converges or diverges

(a)  $\left\{\frac{1}{4}n^2 - 1\right\}$       (b)  $\{(-1)^{n-1}\}$

## L' Hopital's rule

It is a method for computing a limit of form  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$  if

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{\infty}{\infty}$ , then we can use L' Hopital's rule which is defined as  $\lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$ .

## Theorem 2 (properties)

Let  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = K$

- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K$ .
- $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = L \cdot K$ .
- $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{K}$ ,  $K \neq 0$ .
- $\lim_{n \rightarrow \infty} C a_n = C L$ .

### Theorem 3 (Absolute value)

For a seq  $\{a_n\}$ ,  $\lim_{n \rightarrow \infty} |a_n| = 0 \Leftrightarrow \lim_{n \rightarrow \infty} a_n = 0$ .

### Theorem 4 (Geometric seq)

- $\lim_{n \rightarrow \infty} r^n = 0$  if  $|r| < 1$
- $\lim_{n \rightarrow \infty} r^n = \infty$  if  $|r| > 1$

**Example:** Determine whether the following sequences converge or diverge

- (1)  $\{\frac{5n}{e^{2n}}\}$ ,      (2)  $\{(\frac{-2}{3})^n\}$       (3)  $\{(1.01)^n\}$       (4)  $\{\frac{2n^2}{5n^2-3}\}$   
(5)  $\{6(\frac{-5}{6})^n\}$       (6)  $\{8 - (\frac{7}{8})^n\}$       (7)  $\{1000 - n\}$       (8)  $\{\frac{4n^4+1}{2n^2-1}\}$   
(9)  $\{\frac{e^n}{4}\}$ .

### Theorem 5 (Sandwich)

If  $a_n$ ,  $b_n$  and  $c_n$  are sequences such that

- $a_n \leq b_n \leq c_n$  for every  $n$
- $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n$ , then  $\lim_{n \rightarrow \infty} c_n = L$ .

### Theorem 6

A bounded, monotonic sequence has limit.

### Notations

- 1  $-1 \leq \sin(\theta) \leq 1$
- 2  $-1 \leq \cos(\theta) \leq 1$
- 3  $0 \leq \cos^2(\theta) \leq 1$
- 4  $-\frac{\pi}{2} \leq \tan^{-1}(\theta) \leq \frac{\pi}{2}$
- 5  $\cos(\pi n) = (-1)^n$



# Examples

▷ Determine whether the following sequences converge or diverge, if they converge find its limits.

$$(1) \left\{ \frac{\ln n}{n} \right\} \quad (2) \left\{ \frac{\tan^{-1} n}{n} \right\} \quad (3) \{e^{-n} \ln n\} \quad (4) \left\{ \frac{\cos^2 n}{3^n} \right\}$$

$$(5) \left\{ (-1)^{n+1} \frac{1}{n} \right\} \quad (6) \left\{ \frac{\cos n}{n} \right\} \quad (7) \left\{ \frac{n^2}{2n-1} - \frac{n^2}{2n+1} \right\}$$

$$(8) \left\{ \left(1 + \frac{1}{n}\right)^2 \right\} \quad (9) \{n^{1/n}\}$$

**Solution:**