QUANTUM MECHANICS: LECTURE 10

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Abstract

This lecture discusses the angular momentum in quantum mechanics

CONTENTS

THE CLASSICAL ANGULAR MOMENTUM

Recall that the angular momentum for a system is given by :

$$\vec{L} = \sum_i \vec{r}_i \wedge \vec{p}_i \tag{1}$$

With $\wedge$ being the cross (wedge) product between the position $\vec{r}_i$ and linear momentum $\vec{p}_i$ of the ith degree of freedom in the system. For a single particle in 3 D we give a precise definition for the angular momentum :

$$\vec{L} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \tag{2}$$

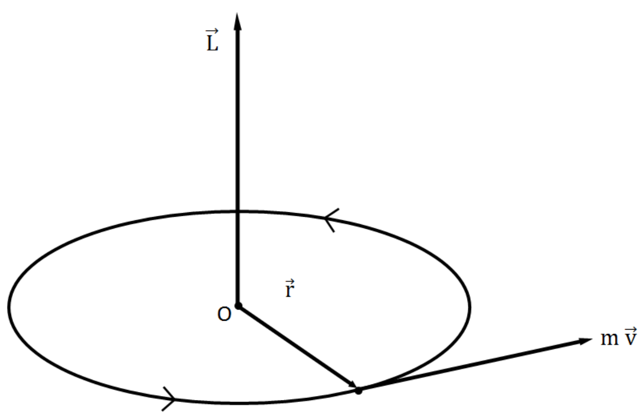

Figure 1: Illustration for the angular momentum of a classical rotating particle

One can apply the canonical quantisation for the angular momentum observable and turn in into a vector operator, since it is defined -classically- by the coordinates and linear momenta :

$$[x^i, p_j] = i\hbar\delta^i_j \qquad [x^i, x^j] = 0 \qquad [p_i, p_j] = 0 \tag{3}$$

Hence the angular momentum operators are (dropping the hat) :

$$\begin{aligned} L_x &= -i\hbar \left(y\partial_z - z\partial_y\right) \\ L_y &= -i\hbar \left(z\partial_x - x\partial_z\right) \\ L_z &= -i\hbar \left(x\partial_y - y\partial_x\right) \end{aligned} \tag{4}$$

Or we may write them in the spherical coordinates (r, φ, θ) :

$$\begin{aligned} L_x &= i\hbar \left(\sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi\right) \\ L_y &= i\hbar \left(-\cos\varphi\partial_\theta + \cot\theta\sin\varphi\partial_\varphi\right) \\ L_z &= -i\hbar\partial_\varphi \end{aligned} \tag{5}$$

We also define the operator :

$$L^2 = L_x^2 + L_y^2 + L_z^2 \tag{6}$$

It is a formidable, yet straightforward task to prove the following commutation relations:

$$[L_x, Ly] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y \tag{7a}$$

$$[L^2, L_i] = 0 \qquad \text{For all } i = x, y, z \tag{7b}$$

We can also define the (rising and lowering) operators:

$$L_\pm = L_x \pm iL_y \tag{8}$$

Along with $L_z = L_3$ they satisfy a well-known commutation relations, known as the $su(2)$ algebra :

$$[L_+, L_-] = 2\hbar L_3 \tag{9a}$$

$$[L_\pm, L_3] = \mp\hbar L_\pm \tag{9b}$$

The rising and lowering operators are expressed in the coordinate representation as :

$$L_\pm = \pm e^{\pm i\varphi} \left(\partial_\theta \pm i\cot\theta\partial_\varphi\right) \tag{10}$$

THE SPHERICAL HARMONICS

As we maintain in the coordinate representation, we wonder about the eigenfunction for the operator L_3 and their properties. The operator L_3 has a special importance over the other two angular momentum operators, as the latter ones compose the ladder(rising and lowering) operators. We start by assuming such wavefunction :

$$L_3 Y_\ell^m(\theta, \varphi) = m Y_\ell^m(\theta, \varphi). \tag{11}$$

With m and ℓ are eigenvalues, and m takes an integer values between ℓ and $-\ell$, this shall be made clear in the next lecture. However, at the meantime, we just accept these as given facts.
Therefore, we conclude that :

$$Y_\ell^m(\theta, \varphi) = e^{im\varphi} y_{\ell m} \tag{12}$$

Moreover, the fact that$L_\pm y_{\ell\pm\ell} = 0$ gives us the differential equation:

$$\left(\partial_\theta - \ell\cot\theta\right) y_{\ell\pm\ell} = 0 \tag{13}$$

Whose complete solution gives us the explicit expression of the functions Y_m^ℓ, which are known as the **Spherical Harmonics**

$$Y_\ell^m = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}} e^{im\varphi} P_m^\ell(\theta) \tag{14}$$

With $P_m^\ell(\theta)$ is the associated Legendre Polynomial.
The spherical harmonics form a complete orthonormal basis for the Hilbert space :

$$\mathcal{H} = \mathcal{L}^2(S^2, d\Omega)$$

With S^2 being the unit sphere, and $d\Omega = d\phi \sin\theta d\theta$, the solid angle element of the unit sphere.

PROPERTIES OF THE SPHERICAL HARMONICS

- Eigenvalue for L^2 :

$$L^2 Y_\ell^m(\theta,\varphi) = \ell(\ell+1) Y_\ell^m(\theta,\varphi). \tag{15}$$

- Orthonormality :

$$\int_{angles} Y_{\ell_1}^{m_1}(\theta,\varphi) Y_{\ell_2}^{*m_2}(\theta,\varphi) d\Omega = \delta_{m_1,m_2}\delta_{\ell_1,\ell_2} \tag{16}$$

- Since the spherical harmonics form an orthonormal basis, the product of two of them is again expressed in terms of spherical harmonics. Take the product $Y_{\ell_1}^{m_1}(\theta,\varphi) \cdot Y_{\ell_2}^{m_2}(\theta,\varphi)$, we can directly conclude that the resultant product is a multiple of the spherical harmonics having $M = m_1 + m_2$ since the term containing m is only an exponential, moreover, L taking the range $|\ell_1 - \ell_2| \leq L \leq ||\ell_1 + \ell_2|$. The general rule for multiplication is given by Wigner $3j$-symbols (or Clebsh-Gordon coefficients) $C(\ell_1, m_1; \ell_2, m_2; L, M)$that shall be studied later in the addition of angular momenta.

$$Y_{\ell_1}^{m_1}(\theta,\varphi) \cdot Y_{\ell_2}^{m_2}(\theta,\varphi) = \sum_{M,L} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2L+1)}{4\pi}} \times \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} Y_L^{*M}(\theta,\varphi) \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Where the Racah symbol is expressed in terms of $3j$ symbol

$$\begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} = (-1)^{\ell_1-\ell_2-M} \frac{1}{\sqrt{2L+1}} C(\ell_1, m_1; \ell_2, m_2; L, -M) \tag{18}$$

These relations will prove useful as we discuss the addition of angular momenta.

- The Herglotz generating function
If the quantum mechanical convention is adopted for the Y_ℓ^m, then,

$$e^{v\mathbf{a}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} \frac{r^\ell v^\ell \lambda^m}{\sqrt{(\ell+m)!(\ell-m)!}} Y_\ell^m. \tag{19}$$

with

$$\mathbf{a} = \hat{\mathbf{z}} - \frac{\lambda}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + \frac{1}{2\lambda}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \tag{20}$$

λ here is a real parameter

More properties are found in the textbooks. We list here some of the spherical harmonics and their graphical representation:

$$Y_0^0(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}} \tag{21}$$

$$Y_1^{-1}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x-iy)}{r} \tag{22}$$

$$Y_1^{0}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \frac{z}{r} \tag{23}$$

$$Y_1^{1}(\theta, \varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x+iy)}{r} \tag{24}$$

$$y = \frac{\sqrt{2x^3(4+2x^2-e^{x^2})^3}}{2x^2 \sin 3x} \tag{25}$$

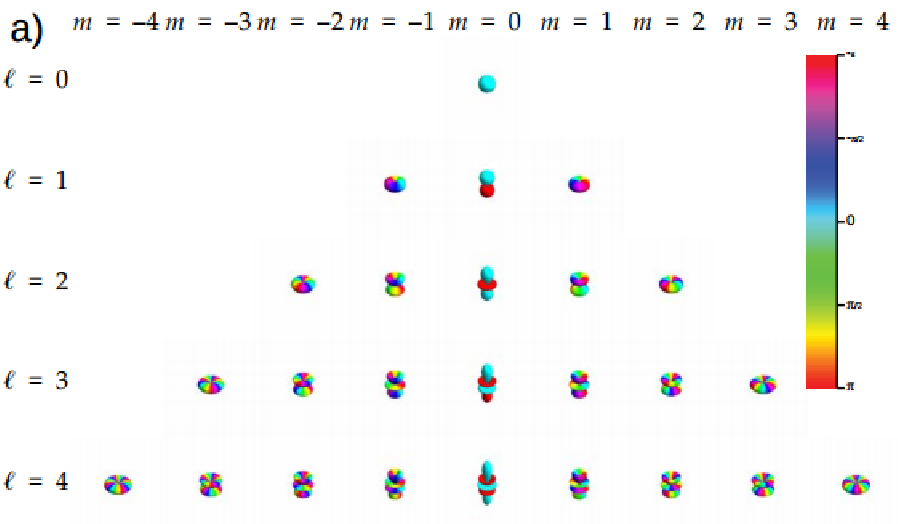

Figure 2: Graphical representation for some of the spherical harmonics, the colour coding represents the probability density calculated via $|Y_m^\ell(\theta, \varphi)|^2$

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