Lecture 11

geometry of continuous groups

topological groups

If G is both a group and a topological space, and if the group operators (the multiplication: $G \times G \to G$ and the inverse: $G \to G$) are continuous, G is a topological group. If G is a topological group, for each $a \in G$, the maps, $L_a : g \to a \cdot g$ and $R_a : g \to g \cdot a$ for $g \in G$ define homeomorphism, called left and right translations.

For a topological group G and its subgroup H, one can define the left and right quotients as follows. For each $g \in G$, define $gH = \{gh : \; h \in H\}$ and call it the left-coset for g. The left quotient G/H is a set of left-cosets. The right quotient is defined by revesing the order of G and H. The quotients are topological spaces. In particular, if H is an invariant subgroup (normal subgroup) of G, namely if, for any $h \in H$ and $g \in G$, ghg^{-1} is also in H, then G/H is also a topological group.

Lie groups

If G is both a group and a differential manifold, and if the group operations are differentiable maps, G is called a Lie group.

Consider the unitary grou $U(N)$ as an example of Lie groups. First we note that any element g of the unitary group can be diagonalized by conjugation by another unitary matrix u as,

$$g = u \cdot \mathrm{diag}(e^{i\theta_a})_{a=1,\dots,N} \cdot u^{-1}.$$

$gg^\dagger = 1$ requires that $\theta_a \in \mathbf{R}$. Since the exponential function has the Taylor expansion,

$$e^{i\theta_a} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \theta_a^n,$$

we can express g as,

$$\begin{aligned} g &= \sum_{n=0}^{\infty} \frac{i^n}{n!} u \cdot \mathrm{diag}(\theta_a^n) \cdot u^{-1} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(u \cdot \mathrm{diag}(\theta_a) \cdot u^{-1} \right)^n \\ &= \exp\left(iu \cdot \mathrm{diag}(\theta_a) \cdot u^{-1} \right). \qquad (1) \end{aligned}$$

Note that $H = u \cdot \mathrm{diag}(\theta_a) \cdot u^{-1}$ is a hermitian matrix. Conversely, if H is a hermitian matrix, $\exp(iH)$ is a unitary matrix. Therefore, any untary matrix can be expressed as an exponential of i times a hermitian matrix.

One important feature of the set of hermitian matrices is that it is closed by the commutator. If H_1 and H_2 are hermitian, then $[H_1, H_2]$ is equal to i times another hermitian matrix. A set of matrices that makes a linear space (linear combinations also belong to the set) and is closed under the commutator in this way is called a Lie algebra.

The need for the closure under the commutator can be explained as floows. Suppose that X and Y are matrices and that we want e^{iX} and e^{iY} to be in the Lie group G. The Baker-Campbell-Hausdorff formula says,

$$e^{iX}e^{iY} = e^{iW},$$

where

$$W = X + Y + \frac{i}{2}[X,Y] - \frac{1}{12}[X-Y,[X,Y]] + \cdots ,$$

and $\cdots$ in the above is expressed as a sum of commutators of X and Y. Thus, if X and Y are in the Lie algebra, so is W. If e^{iX} and e^{iY} belong to G, so is $e^{iX}e^{iY}$.

Pull-back, Push-forward, Lie derivative

These concepts are defined for any differentiable manifold, but they are useful to study geometry of Lie groups.

Suppose there is a differentiable map, $\varphi : M \to N$, between manifolds M and N. We do not have to assume φ is injective. This map induces the pull-back, $\varphi^* : C^\infty(N) \to C^\infty(M)$. For any function $f(q)$ $(q \in N)$, we can defined a function on M as $[\varphi^* f](p) = f(\varphi(p))$ $(p \in M)$.

We can also define the push-forward tangent vector fields on M and N. Pick a tangent vector v on M. For any differentiable function f on M, it gives another function $v(f)(p)$. What we want is a way to find a function on N for each function g on N. This can be done as follows. First pull-back g to define a function $[\varphi^* g](p) = g(\varphi(p))$ on M. Now we can evaluate it with v to define another function $v(\varphi^* g)(p) = v(g(\varphi(p)))$. Thus, the tangent vector $v(p)$ at T_pM is mapped to $[\varphi_* v](q)$ at T_qN, where $q = \varphi(p)$.

Question 1: This may sounds a bit abstract, so let us express it in terms of coordinates. Choose coordinates x^μ $(\mu = 1, ..., m)$ on M and y^i $(i = 1, ..., n)$ on N. Note that M and N may have different dimensions! Consider the tangent vector field $v = v^\mu(x)\partial/\partial x^\mu$. For $\varphi : x^\mu \to y^i(x)$, show

$$\varphi_* v = v^\mu \frac{\partial y^i}{\partial x^\mu}\frac{\partial}{\partial y^i}.$$

In the GR speak, the tangent vector v transforms as a contravariant vector.

We can repeat this for one-forms. Since a one-form is a linear function on tangent vectors, we can naturally define a pull-back φ_*.

Question 2: Define a pull-back for a one-form ω. Show that, in coordinates,

$$\omega = \omega_i dy^i \to \varphi^*\omega = \omega_i(\varphi(x))\frac{\partial y^i}{\partial x^\mu}dx^\mu.$$

Now we define a Lie derivative for a given tangent vector field v. For a given vector field v on M, we can define the exponential map, $\varphi_t : M \to M$, defined by the property that

$$\frac{d}{dt}f(\varphi_t(p)) = [v(f)](\varphi_t(p)).$$

We often write it as $\varphi_t(p) = \exp(tv)(p)$. Infinitesimally (which is all we need for now), and in terms of coordinates x^μ,

$$\varphi_t(x)^\mu = x^\mu + tv^\mu(x) + \cdots .$$

Now, we can define a push-forward map, $\varphi_{-t\ *} : T_{\varphi_t(p)}M \to T_pM$. Using this, the Lie derivative $\mathcal{L}_v$ is defined as a map from a tangent vector field u on M to another tangent vector field $\mathcal{L}_v u$,

$$\mathcal{L}_v u(p) = \lim_{t\to 0}\frac{1}{t}\left(\varphi_{-t\ *} v(\varphi_t(p)) - v(p)\right).$$

Question 3: Writing u and v as differential operators $u = u^\mu\partial_\mu$, $v = v^\mu\partial_\mu$, show,

$$\mathcal{L}_v u = [v, u].$$

Using coordinates, show that $[v, u]$ transforms as a tangent vector field.

Lie Algebra as Space of Left-Invariant Vector Fields

Cnsidering the Lie group G as a differential manifold, the left and right translations of $a \in G$,

$$L_a : g \to ag, \qquad R_a : g \to ga,$$

define diffeomorphisms from G to itself. Thus, we can define push-forwards, L_{a*} and R_{a*}. We say that a tangent vector field v is left (right) invariant iff $L_{a*}v = v$ ($R_{a*}v = v$). Let us denote the space of left invariant vector fields of G by $\mathcal{G}$. If v is a left invariant vector field, $v(g) = L_{a*}v(a^{-1}g) = L_{g*}v(1)$. Thus, it is determined by a tangent vector at $g = 1$. This means that $\mathcal{G}$ as a linear space is isomorphic to $T_{g=1}G$. In particular, dim $\mathcal{G}$ = dim G.

If u and v are left invariant vector fields, so is $[u, v]$. This way, the space $\mathcal{G}$ of left invariant vector fields can be identified with the Lie algebra of G.

If we think of the group G as a matrix group, with each element $g \in G$ represented by a matrix g_{ij}, left invariant vector fields can also be written as

$$\sum_k g_{ki}\frac{\partial}{\partial g_{kj}}.$$

For each (i, j), the above vector is invariant under $g \to ag$.

Maurer-Cartan Forms

The Maurer-Cartan forms are left invariant one-forms, defined by

$$\Phi = g^{-1}dg.$$

It satisfies the Maurer-Cartan equation,

$$d\Phi + \Phi\wedge\Phi = 0.$$

The left invariant vector field $g^T\partial/\partial g$ is dual to the Maurer-Cartan forms. In fact, the Maurer-Cartan form is locally determined by this equation since we can think of the equation as saying that the gauge connection one-form given by Φ has vanishing curvature and is locally gauge equivalent to the trivial configuration.

Connected and Simpy Connected

Two elements of a Lie Group G are called connected if there is a continuous path in the group connecting them. Being connected is an equivalence relation, and thus the group can be divided into its equivalence classes, called connected components. Let us call the connected component containing the identity e as G_0. Since any element of the unitary group $U(N)$ can be expressed as $\exp(iX)$ by some hermitian matrix X, the unitary group is connected.

It is easy to see that G_0 is a sugroup of G. It important to note that G_0 is an invariant subgroup (normal subgroup). Namely, for any $g \in G$ and $h \in G_0$, ghg^{-1} is also in G_0. To see this, suppose $h(t)$ is a continuous path connecting e to h as $h(0) = e, h(1) = h$. It then follows that $gh(t)g^{-1}$ gives a path connecting e to ghg^{-1}.

If $\pi_1(G_0)$ is trivial, G_0 is called simply connected.

examples

$O(3)$ is not connected. To see that, we note that any $g \in O(3)$ satisfies $gg^T = 1$ and thus $(\det g)^2 = 1$. Thus, elements of $O(3)$ are divided into those with $\det g = 1$ and -1, and these two classes of elements are not continuously connected. On the other hand, $SO(3)$ is connected. However, it is not simply conneced. As explained below, $SU(2)$ is locally the same as $SO(3)$, but it is both connected and simply connected.

Universal Covering Group

The fundamental theorem of the Lie group (initiated by S. Lie and completed by E. Cartan) states that, for any n-dimensional Lie algebra, there is a unique simply connected Lie group. This simply connected group is called the universal coversing group and is denoted by UG. Any other Lie group with the same Lie algebra is of the form UG/Γ, where Γ is a discrete invariant subgroup.

In fact, if Γ is a discrete invariant subgroup, each of its elements must commutes with elements of UG. To see this, note that Γ being invariant means that, for any $h \in \Gamma$ and for any $g \in UG$, there is $h' \in \Gamma$ such that $ghg^{-1} = h'$. However, g can be continuously connected to the identity e in UG. Since Γ is discrete, h and h' cannot change continuously as we change g to e. Thus, $ehe^{-1} = h = h'$, and $gh = hg$.

If we regard G as a matrix group, then by Shur's lemma, any group element that commutes with all the elements of G must be proportional to the identity. Therefore, Γ should consist of elements of the form $\lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbf{C}$.

examples

Let us enumerate all discrete invariant subroups of $SU(2)$. The group $SU(2)$ consists of matrices of the form,

$$g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}.$$

The condition that $\det g = 1$ means

$$|\alpha|^2 + |\beta|^2 = 1.$$

Namely, the group $SU(2)$ is diffeomorphic to the 3-sphere in $\mathbf{R}^4$. If Γ is an invariant subgroup, its element must be of the form $\lambda \cdot \mathrm{id}$. For this to belong to $SU(2)$, $\lambda = \pm 1$.

Thus, there are two possibilities: $\Gamma = \{\mathrm{id}\}$ or $= \{\pm\mathrm{id}\}$. In the former case, we have $SU(2)$ itself. In the latter case, we have $SU(2)/\mathbf{Z}_2$. The $\mathbf{Z}_2$ action identifies $\alpha \sim -\alpha$, $\beta \sim -\beta$. This gives the group $SO(3)$. To see that, we note that any rotation in three dimensions can be parametrized by the vector $\vec{\Omega} = \theta\vec{n}$, where $\vec{n}$ is a unit vector representing the axis of rotation and θ is the amount of rotation. We can choose, for example, $0 \leq \theta \leq \pi$. Thus, the space of $(\theta, \vec{n})$ is the disk, with $\theta = \pi$ representating the boundary of the disk. However, we need to identify $(\pi, \vec{n})$ with $(\pi, -\vec{n})$. Namely, we make the antipodal identification of the boundary of the disk. If we view the disk as the upper hemisphere of the S^3, it is the same as $SU(2)/\mathbf{Z}_2$.

In general, the universal covering group of $SO(N)$ is called $spin(N)$. For example, $spin(3) \sim SU(2)$, $spin(4) \sim SU(2) \times SU(2)$, $spin(5) \sim USp(4)$, $spin(6) \sim SU(4)$.

Homotopy of Lie Groups

If $\pi_1(G)$ trivial, $\pi_1(G/H) = \pi_0(H)/\pi_0(G)$. Thus, for example, we see that $\pi_1(SO(3)) = \pi_1(SU(2)/Z_2) = \pi_0(Z_2) = Z_2$.

If $\pi_2(G)$ is trivial, $\pi_2(G/H) = \pi_1(H)/\pi_1(G)$. For example, for any compact connected Lie group, $\pi_2(G)$ is trivial. In a gauge theory with the Higgs mechanism to break the gauge symmetry $G \to H$, in the symmetry breaking phase, the Higgs field Ψ takes value in G/H. Suppose we are in $\mathbf{R}^3$ and there is a point in the space where the gauge symmetry is restored. There, the Higgs field must vanish. So, we remove the neighborhood of the point from $\mathbf{R}^3$. The resulting space is homotopic to S^2. Thus, the Higgs field configuration is classified by $S^2 \to G/H$. This gives the classification of the 't Hooft-Polyakov magnetic monopole.

In the Standard Model of Particle Physics , the gauge group $SU(3) \times SU(2) \times U(1)$ is spontaneously broken to $SU(3) \times U(1)_{\rm EM}$, where $U(1)_{\rm EM}$ is a combination of the $U(1)$ and an $U(1)$ subgroup of the $SU(2)$ in the original gauge group. Thus, we should consider $\pi_2(SU(2) \times U(1)/U(1)_{\rm EM}) = \pi_1(U(1)_{\rm EM})/\pi_1(U(1))$ and it is trivial, where we used the fact that the $SU(2)$ is simply connected. This shows that there is no magnetic monopole with the Standard Model.

However, in the grand unified theory with gauge groups $SU(5)$ or $SO(10)$, we can have $\mathbf{Z}$ valued magnetic monopoles since $\pi_1(SU(5))$ is trivial and $\pi_1(SO(n)) = \mathbf{Z}_2$ $(n \geq 3)$. This is a prediction of these grand unified theories.

To prove these relations, we can use the exact sequence,

$$\begin{aligned} &\to \pi_3(G) \to \pi_3(G/H) \to \pi_2(H) \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \\ &\quad \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(G) \to \pi_0(G) \to \pi_0(G/H) \to 0. \end{aligned} \tag{2}$$

That this is the exact sequence means that the image of each map is the kernel of the next-right map. For example, the image of $\pi_1(G)$ in $\pi_1(G/H)$ is the kernel of the map from $\pi_1(G/H)$ into $\pi_0(G)$. Thus, if $\pi_1(G)$ is trivial, there is no kernel for the map from $\pi_1(G/H)$ to $\pi_0(G)$, i.e., it is injective. Thus, $\pi_1(G/H)$ must be the same as its image in $\pi_0(G)$. But, by the exact sequence, the image must be the same as the kernel of the map from $\pi_0(G)$ to $\pi_0(H)$. The latter is the same as the quotient $\pi_0(H)/\pi_0(G)$. This shows that $\pi_1(G/H) = \pi_0(H)/\pi_0(G)$. Similar relations can be derived for $\pi_n(G/H)$.