QUANTUM MECHANICS: LECTURE 13

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Abstract

This lecture first defines the external and internal degrees of freedom, the mathematical formulation of internal degrees of freedom and their quantisation. This lecture provides a basis for the philosophy of representation theory, that lies the foundation of Fundamental physics.

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TRANSFORMATION OF THE WAVEFUNCTION

Given a wavefunction defined at the point (in 1-D) x_0, then we define the **infinitesimal translation** of the wave function as a transformation preformed on it, by shifting the point x_0 by an infinitesimal parameter ϵ.

$$\psi(x_0) \longrightarrow \psi(x_0+\epsilon)$$

Using the Taylor series expansion of the translated wavefuntion around the point x_0 we can write :

$$\psi(x_0+\epsilon) = \psi(x_0) + \epsilon\frac{d\psi}{dx} + \mathcal{O}(\epsilon^2). \tag{1}$$

since ϵ is an infinitesimal parameter , the order terms of ϵ^2 are considered vanishing, hence:

$$\psi(x_0+\epsilon) \sim \psi(x_0) + \epsilon\frac{d\psi}{dx} \tag{2}$$

It won't affect the expansion if we multiplied and divided by $\frac{i}{\hbar}$:

$$\psi(x_0+\epsilon) \sim \psi(x_0) + \epsilon\,\frac{i}{\hbar}\,\left(\frac{\hbar}{i}\frac{d\psi}{dx}\right), \tag{3}$$

we hereby identify $\frac{\hbar}{i}\frac{d}{dx}$ as the m operator, $\hat{P}$. Thus,

$$\psi(x_0+\epsilon) \sim \psi(x_0) + \epsilon\,\frac{i}{\hbar}\,\left(\hat{P}(\psi)\right). \tag{4}$$

This equation basically tells us that the translation is **generated** by the momentum operator , or the momentum operator is the **generator of translation**.
Same argument can be made for 3-D, in Cartesian coordinates $\vec{r} = (x, y, x_{)}$, being translated by $\vec{\delta r}$. That is,

$$\psi(\vec{r}) \longrightarrow \psi(\vec{r}+\vec{\delta r})$$

By the same argument done before, the multivariate Tylor expansion :

$$\psi(\vec{r}+\vec{\delta r}) \sim \delta r \, \frac{i}{\hbar} \, \left(\frac{\hbar}{i} \nabla(\psi) \right). \quad (5)$$

We know that the 'linear momentum operator is defined by $\vec{P}\frac{\hbar}{i}\nabla$, where the gradient operator here is in the Cartesian coordinates .
Now, we can run the same argument for multi-particle system with f number of degrees of freedom in generalised coordinates : $(q^1, q^2, \ldots q^f)$, the transformation of generalised coordinates is not restricted translations, but also rotations (if some of the generalised coordinates correspond to angles for example.). But such transformation is written in the form : $\frac{\partial}{\partial q^i}$, that is, it depends on the derivatives of the coordinates - for a general configuration space- .

external and internal degrees of freedom for a system

The f degrees of freedom the system has, can be linked to a set of generalised coordinates of the configuration space. And the transformation of the wavefunction for that system's degrees of freedom is written in terms of (derivatives of the generalised coordinates), or we can say that such transformations are translations in space and time, in addition to 3-D rotations.
These degrees of freedom are called **external** degrees of freedom. The observables that are explicit functions of generalised coordinates are linked to these degrees of freedom. In other words, the operators corresponding to those observables act - effectively- on the same Hilbert space.

Examples : The linear and angular momentum, the velocity and energy...

However, in quantum mechanics, there exist another type of degrees of freedom, that do not (explicitly) depend on space and time. (or configuration space). Rather they form a separate Hilbert space, with operators correspond to observables that are not an explicit functions of generalised coordinates, nor time. Although they can affect the external degrees of freedom observables, through physical process called **coupling**. But this is only when we consider the product of the two Hilbert spaces - the system as a whole-.
This is better understood via the transformations we have discussed earlier. As a space/ configuration space transformation is preformed, the internal degrees of freedom of the system are not affected by themselves. This is the main difference between internal and external degrees of freedom of a system.
In fact, there is a theorem in mathematical physics called the *Coleman Mandula theorem,* that separates space and time translation symmetries and internal ones (one cannot be obtained from the other.) unless *supersymmetry* is used, this is beyond the scope of our study.
The distinction between, scape and time (explicitly) dependent observables, and internal degrees of freedom is crucial in the understand of elementary particle physics in one hand or quantum information in the other. There are many examples for observables originating from internal degrees of freedom of quantum particles. . Electric charge, magnetic dipole moment (from free electron), quantum numbers like: leptonic or baryonic quantum numbers, strangeness...etc

We should emphisise that they are degrees of freedom and not internal structure , as (elementary) particles are as-far-as we know dimentionless .

mathematical description for internal degrees of freedom

As we mentioned above, the internal degrees of freedom for a quantum system has an independent Hilbert space that we dealt with earlier (we can always go to L^2 for external ones). Now, we need to study how such space can be defined, or constructed.

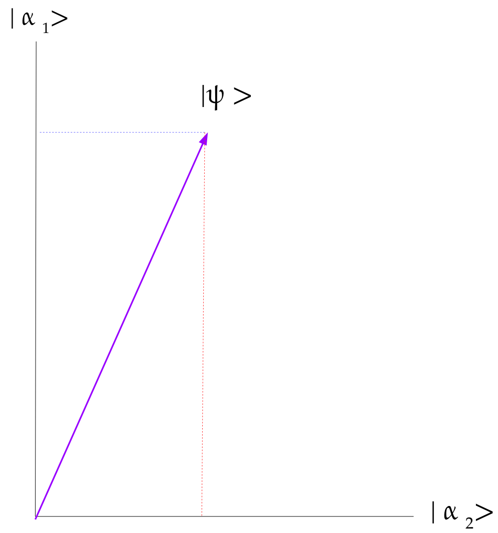


Figure 1: The quantum state can be abstractly represented as a vector in the state space

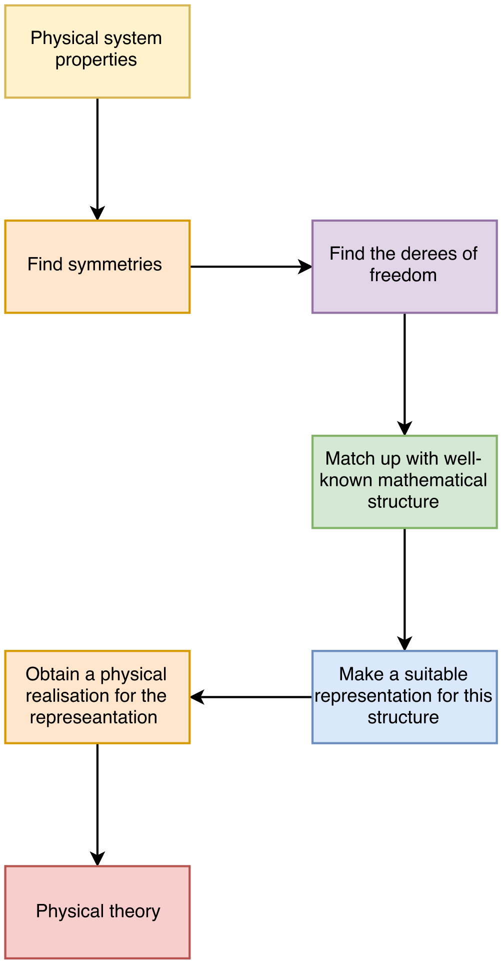
We start by defining the state ket $|\psi$, decomposed into the basis for the Hilbert space $\{|\alpha_i\rangle\}$.

$$|\psi\rangle = \sum_{i=1}^{\dim \mathcal{H}} \alpha_i |\alpha_i\rangle \tag{6}$$

Such that, the dimension of the Hilbert space is the same as the number of the internal degrees of freedom described by that space. The coefficients α_i are the probability amplitudes of detecting the system having the state α_i, i.e. usually the quantum system is in a state of superposition, until it is measured. Identically to what is studied before.
We can picture a state space for these degrees of freedom and define transformations just like we have done in the external ones 3 Since the magnitude of $|\psi\rangle$ should be 1, i.e.

$$||\psi\rangle| = 1,$$

any transformation made upon this ket is ought to a rotation in the state space. If that space is quantised, the rotation is only possible in integer multiples , and via a ladder (raising and lowering) operators similar to what we have seen in the (orbital) angular momentum. Hence all these internal degrees of freedom are **physical realisations** of the same mathematical structure discussed earlier in the angular momentum.
These facts are very powerful, because it allows us to apply the same techniques, and thought processes to many physical systems (with modifications that appears naturally). Depending on the nature of the physical problem (the symmetry, the nu,ber of degrees of freedom ..etc). To illustrate this power, this technique (which is known formally as **representation theory**) is used extensively in Fundamental physics. The whole standard model of particle physics is fundamentally built upon the same idea of *representation*, quantum electrodynamics and quantum chromodynamics ..etc


Physical system properties
Find symmetries
Find the derees of freedom
Match up with well-known mathematical structure
Make a suitable representation for this structure
Obtain a physical realisation for the represeantation
Physical theory