QUANTUM MECHANICS: LECTURE 14

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Abstract

Introduction to the electron's spin from the electron dipole moment and construction of commutation relations of the spin operator . Van de Warden symbols (Pauli spin matrices), and their mathematical properties.

CONTENTS

1 DISCOVERY OF ELECTRON'S MAGNETIC DIPOLE MOMENT

In 1925, the idea of 'spin' angular momentum for electron was first proposed George Uhlenbeck and Samuel Goudsmit to explain hyperfine splitting in atomic spectra. However, in 1922, Otto Stern and Walther Gerlach had shown that electrons act like tiny magnetic bars (they have a magnetic dipole moment), and that magnetic moment takes particular values only.
We are going to construct step-by-step the theory of quantum spin via the observations of S-G experiment, and discuss the mathematical properties of our construction. Following the lead of last lecture's philosophy.

S-G apparatus is basically a magnetic field that is free to be in any direction, x, y, z. As a beam of electrons is passed through the S-G apparatus, the beam splits either in the positive or negative direction (say the B field is aligned with the z axis). The split is due to a change of energy of the beam, because electron's have magnetic dipole moment, and it takes a $\pm$ values or $1/2$ values only, otherwise ... If the electron has an integer multiples of some magnetic dipole moment, not $1/2$ the splitting would be at least in three ways, for $\pm$ direction and remaning of the beam passing through unaffected. Quantitatively we write the change of the energy is given by :

$$\Delta H = \vec{\mu} \cdot \vec{B}. \tag{1}$$

Since $\vec{B} = B_z$ only the projection of the magnetic dipole moment in the z direction counts, we hence conclude that :

$$\mu_z = \pm\frac{1}{2} \quad \cdot const. \tag{2}$$

Same effect is observed if the B field was aligned with any axis not just the z axis. Hence the observable of the dipole magnetic moment is associated with a vector operator $\vec{\hat{\mu}}$ that has an eigenvalues of $1/2$ of some constant. This constant was found experimentally, and called the **Born magnetron** μ_B for the electron, but generally it is noted by γ the **gyromagnetic ratio**, being more careful, Born magnetro and the gyromagnetic ratio are not exactly equal, due to relativistic effects, and Thomas precession $\gamma = g_s \cdot \mu_B$ where g_s

is **called Landé g-factor** .
In fact we write :

$$\hat{\vec{\mu}} = -g_s \mu_B \frac{\hat{\vec{S}}}{\hbar} \tag{3}$$

We call the operator $\hat{\vec{S}}$ the spin operator .

2 ELECTRON'S SPIN

Although $\hat{\vec{S}}$ does not correspond to a direct observable (nobody can detect the spin directly separately from the magnetic dipole moment) But seems very natural to study the spin rather than the magnetic dipole moment directly. Although there is no classical analogy to the spin, as electrons do not spin around themselves (they are dimensionless points) but spin is a form of angular momentum corresponding to an internal degree of freedom for the electron.
So far, from S-G experiment, we can conclude that :

$$\langle \hat{S}_z \rangle = \pm \frac{1}{2}\hbar \tag{4}$$

We hall stick with z but any component gives as similar results as the S_z. And the $\hbar$ is just to reserve the dimension of the spin, since it is an angular momentum. We let , for convince denote :

$$\langle \hat{S}_z \rangle = m_s \qquad m_s = -\frac{1}{2}\hbar, +\frac{1}{2}\hbar \tag{5}$$

Now, we turn into studying the spin operators more, via repeating the S-G experiment in the following way:
Start with S-G apparatus in the z direction, take the half of the initial beam that is aligned in the $+z$ direction (having $m_s = +1/2\hbar$) passed though another S-G apparatus but aligned in the x direction, it shall split the beam once again. This time in the $\pm x$ directions. One would expect that by these two apparatuses; we were able to measure two observables associated with $\hat{S}_z$ and $\hat{S}_x$. Nevertheless, experiments have shown that this is not true, as taking the $1/4$ of the beam that passed via the $+z$ first and $+x$ second, to a third S-G apparatus aligned in the z. We expect only one beam to pass in the $+z$ direction, but this doe not happen. The beam splits into two a third time ! This does not happen if we passed it (initially) into three consecutive S-G apparatuses aligned in the z direction. Hence, we cannot measure the spin in to direction simultaneously. This is mathematically written as :

Note how similarities in notation with the orbital angular momentum is appearing

$$[\hat{S}_z, \hat{S}_x] \neq 0 \tag{6}$$

In fact the commutation relation for the spin operators take the form for the indices i, j, k taking the values x, y, z:

$$[\hat{S}_i \hat{S}_j] = i\hbar \epsilon_{ij}^k \hat{S}_k \tag{7}$$

Which is the same as for the $\hat{L}$'s that we studied before. We can therefore, adopting the philosophy of previous lectures define the following operators:

The symbol ϵ_{ij}^k is called the Levi-civita symbol and it is equal to 0 if $i = j$ and $+1$ for even permutation and -1 for odd permutations of the indices

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y \tag{8}$$

and:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \tag{9}$$

With the eigenstates :

$$|s, m_s\rangle \tag{10}$$

But the eigenvalue s take one value only $s = \frac{1}{2}\hbar$ and m_s as we have seen takes the values $m_s = \frac{\pm 1}{2}\hbar$:

$$\begin{aligned} \hat{S}_z|s, m_s\rangle &= m_s\hbar|s, m_s\rangle \\ \hat{S}|s, m_s\rangle &= \hbar\sqrt{s(s+1)}|s, m_s\rangle \end{aligned} \tag{11}$$

And the ladder operator :

These operators are a direct result from the quantisation of the spin space

$$\hat{S}_{\pm}|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}|s, m_s \pm 1\rangle \tag{12}$$

The three operators $\hat{S}_{\pm}$ and $\hat{S}_z$ form the well-known $su(2)$ algebra commutation relations:

$$\begin{gathered}[\hat{S}_{\pm}, \hat{S}_z] = \mp\hbar\hat{S}_{\pm} \\ [\hat{S}_+, \hat{S}_-] = 2\hbar\hat{S}_z\end{gathered} \tag{13}$$

The Hilbert space for the spin is very simple, as it is 2-D only spanned by the basis $|\frac{1}{2}, +\frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle$ or denoted by $|\chi_+\rangle, |\chi_-\rangle\rangle$, respectively.

$$|\psi\rangle = \sum_{\lambda=\pm} \alpha_\lambda |\chi_\lambda\rangle \tag{14}$$

representing a superposition of the spin states.

3 Infel-van der Warden symbols

Not always we can have the luxury of selecting the Cartesian coordinates in order to study the spin alignment, sometimes we need to work with any suitable set of coordinates (cylindrical, spherical . . .) . Hence we need to link the spin operator vector to a more mathematically-rigorous argument other than the observations made from S-G experiment.

In fact, we may write the spin operator vector $\hat{\vec{S}}$ in terms of a set of symbols, known as **Infel-van der Warden Symbols** ($\sigma^1, \sigma^2, \sigma^3$):

Usually Infel-van der Warden Symbols used in advanced texts differ from our notation by the factor of i, i.e. they usualy call $i\sigma^i$ as the symbol

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma} \tag{15}$$

Where, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is called the **Pauli vector** , and these symbols are known in the physics literature as **Pauli spin matrices** , Due to their relation - in this context- with the spin operator.

It goes without saying that the spin operators really have inherited the mathematical properties (commutation relations) from the Infel-van der Warden Symbols, or -as we shall call them from now on- the Pauli spin matrices. So, naturally there exist the two symobols:

$$\sigma^{\pm} = \sigma^1 \pm i\sigma^2 \tag{16}$$

That along with σ^3 form the $su(2)$ algebra commutation relations:

$$\begin{gathered}[\sigma^3, \sigma^{\pm}] = \pm i\sigma^{\pm} \\ [\sigma^+, \sigma^-] = i\sigma^3\end{gathered} \tag{17}$$

Also note that

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ij}_k \sigma^k$$

Moreover, the pauli spin matrices have an important mathematical property called the **Clifford algebra** relation:

$$\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij} \tag{18}$$

We call the operation $\{\cdot, \cdot\}$ the **anticommutator**. From this property we can easily prove that:

$$(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \delta^{ij} \tag{19}$$

So far, we only dealt with *abstract*, mathematical entities, and their properties (commutation relations), In order to use them in physical calculations, we need to find a proper **representation** for them in order to be realised in the physical world. We shall find that there are many possible representations for the Pauli spin matrices, that have a direct physical application and meaning. In fact, spin of the electron is only a simple application of the representations for the Pauli spin matrices, others go as deep as the *electroweak interaction* in the standard model of particle physics, and supersymmetry! However, there are other representations, that seems to be of an interest of mathematicians mainly like the quaternions (a higher form of complex numbers that has 3 imaginary units) .