QUANTUM MECHANICS: LECTURE 1

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Abstract

In this lecture, a revision of basic concepts of classical Hamiltonian dynamics. Such concepts will play an important rôle in understanding quantum mechanics, and the quantisation of classical system.

CONTENTS

THE HAMILTONIAN

We have seen the **Hamiltonian** function in classical mechanics; resembling the energy of the dynamical system in study.

recall that $H = T + V$

$$H(\vec{p}, \vec{q}) \equiv \text{ Energy of the system.} \tag{1}$$

Where, $\vec{p}$ and $\vec{q}$ are the generalised momentum and coordinates(configuration) for the system. We define the generalised/ canonical momentum as:

Note that $\vec{p}$ here is not always $m\vec{v}$!

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{2}$$

With $\mathcal{L}$ being the Lagrangian function.

recall that $\mathcal{L} = T - V$

Examples

1. The Hamiltonian for a free particle is given by:

$$H = \frac{p^2}{2m} \tag{3}$$

2. In a central potential, the Hamiltonian takes the form :

$$H = \frac{p^2}{2m} + g\,\frac{Q_1\,Q_2}{r} \tag{4}$$

   Where g is a constant, called *coupling constant*, and $Q_1\,Q_2$ are charges/ masses. Depending on the type of interaction.

3. A dipole in a magnetic field B has the following Hamiltonian:

$$H = \vec{\mu} \cdot \vec{B} \tag{5}$$

4. The Hamiltonian for a free rotating mass is:

recall that $I = Mr^2$

$$H = \frac{L^2}{2I} \tag{6}$$

Here, L is the angular momentum of the system.

5. Lastly, a very important Hamiltonian, is the Hamiltonian for a simple harmonic oscillator (SHO).

$$H = \frac{P^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{7}$$

hamilton's equations

For a dynamical system with a Hamiltonian. The evolution of that system obeys the set of equations , known as **Hamilton's equations** :

note that both $\vec{p}$ and $\vec{q}$ are functions of time

$$\frac{\partial H}{\partial q^i} = -\dot{p}_i \tag{8}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{9}$$

Some dynamical systems however, does not obey these equations. These systems are known as *Hamiltonian constraint* systems.

Poisson brackets

Let $f(\vec{q}, \vec{p})$, and $g(\vec{q}, \vec{p})$ be functions of both $\vec{q}$ and $\vec{p}$. One may define the following operation :

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) \tag{10}$$

This operation is called **Poisson brackets**, it shall prove importance in quantisation, and also in understanding conserved quantities (see homework). It is interesting to Poisson bracket the Hamiltonian with components of $\vec{q}$ and $\vec{p}$, for example :

$$\begin{aligned} \{q^j, H\} &= \sum_i \frac{\partial q^j}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \underbrace{\frac{\partial q^j}{\partial p_i}}_{=0} \\ &= \sum_i \delta^j_i \underbrace{\frac{\partial H}{\partial p_i}}_{=\dot{q}^i} \\ &= \dot{q}^j \end{aligned} \tag{11}$$

Same goes for:

$$\{p_j, H\} = \dot{p}_j \tag{12}$$

This actually lead us to the general result, for any function $f(\vec{q}, \vec{p})$, we have:

$$\{f(\vec{q}, \vec{p}), H\} = \frac{\partial f(\vec{q}, \vec{p})}{\partial t} \tag{13}$$

phase space

Since one only needs the p's and q's in order to full describe the dynamical system. The **state** vector is defined in $2N$ dimensional space, called the phase space.

One needs as many of them as the number of degrees of freedom N for the system

$$\vec{S}(t) \equiv (\vec{q}(t), \vec{p}(t)) \tag{14}$$

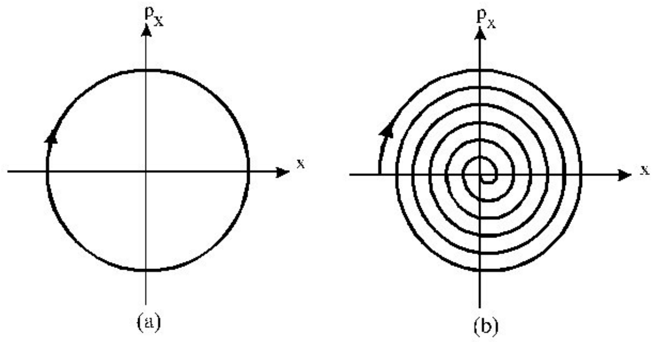

Figure 1: (a) The phase space for free SHO. (B) The phase space of a damped SHO

Example: phase-space for SHO

This phase space for the SHO in 1-D is resembled by a circle: The circle resembles all the possible states the SHO can take, at any moment in time. In other words, this describes the full evolution of the system in time.
Since energy is conserved for the free SHO, the system will keep being in the circle. As for damped one, when energy is lost. The system will spiral until it comes to a halt.