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Pure birth process

Lecture (22)

For pure birth process, If

$$P_0'(t) = -\lambda_0 P_0(t) \qquad (1)$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \quad \text{for } n \geq 1 \qquad (2)$$

with initial Condition $X(0) = 0$

or $P_n(0) = \begin{cases} 1 & , n = 0 \\ 0 & , \text{otherwise} \end{cases}$

where $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ are the birth parameters

then $\boxed{P_0(t) = e^{-\lambda_0 t}}$ (3)

$$P_1(t) = \lambda_0 \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right] \qquad (4)$$

and $P_n(t) = pr\{X(t) = n \mid X(0) = 0\}$ (5)

$$= \lambda_0 \lambda_1 \cdots \lambda_{n-1} \left[B_{0,n} e^{-\lambda_0 t} + \cdots + B_{k,n} e^{-\lambda_k t} + \cdots + B_{n,n} e^{-\lambda_n t} \right], \quad n > 1$$

where

$$B_{k,n} = \prod_{i=0}^{n} \left(\frac{1}{\lambda_i - \lambda_k} \right) \quad i \neq k, \quad 0 < k < n \qquad (5-1)$$

$$, B_{0,n} = \prod_{i=1}^{n} \left(\frac{1}{\lambda_i - \lambda_0} \right) \qquad (5-2)$$

$$, B_{n,n} = \prod_{i=0}^{n-1} \left(\frac{1}{\lambda_i - \lambda_n} \right) \qquad (5-3)$$

See, p 280, 281 Textbook

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pb 6.1.1 p.283 Text

A pure birth process starting from $X(0)=0$ has birth parameters $\lambda_0=1$, $\lambda_1=3$, $\lambda_2=2$ and $\lambda_3=5$

Determine $P_n(t)$ for $n=0,1,2,3$

Ans

$\boxed{\text{at } n=0}$

$$(3) \Rightarrow P_0(t) = e^{-\lambda_0 t}, \quad \lambda_0=1$$

$$\therefore \boxed{P_0(t) = e^{-t}}$$

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$\boxed{\text{at } n=1}$

$$(4) \Rightarrow P_1(t) = \lambda_0\left[\frac{1}{\lambda_1-\lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0-\lambda_1} e^{-\lambda_1 t}\right]$$

$\lambda_0=1,\ \lambda_1=3$

$$P_1(t) = 1\left[\frac{1}{3-1} e^{-t} + \frac{1}{1-3} e^{-3t}\right]$$

$$\therefore P_1(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} = \frac{1}{2}\left[e^{-t} - e^{-3t}\right]$$

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$\boxed{\text{at } n=2}$

$$(5) \Rightarrow P_2(t) = pr\{X(t)=2 \mid X(0)=0\}$$

$$= \lambda_0\lambda_1\left[B_{0,2}\, e^{-\lambda_0 t} + B_{1,2}\, e^{-\lambda_1 t} + B_{2,2}\, e^{-\lambda_2 t}\right] \quad \text{I}$$

$$(5\text{-}2) \Rightarrow B_{0,2} = \prod_{i=1}^{2}\left(\frac{1}{\lambda_i-\lambda_0}\right)$$

$$B_{0,2} = \frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)} = \frac{1}{(3-1)(2-1)}$$

$$B_{0,2} = \boxed{\frac{1}{2}}$$

$\lambda_0=1$
$\lambda_1=3$
$\lambda_2=2$

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$$(5\text{-}1) \Rightarrow B_{1,2} = \prod_{i=0}^{2} \frac{1}{\lambda_i - \lambda_1}, \quad i \neq 1$$

$k \quad n$

$\lambda_0 = 1$
$\lambda_1 = 3$
$\lambda_2 = 2$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$$B_{1,2} = \frac{1}{(1-3)(2-3)} = \boxed{\frac{1}{2}}$$

$$(5\text{-}3) \Rightarrow B_{2,2} = \prod_{i=0}^{1} \frac{1}{\lambda_i - \lambda_2}$$

$n \quad n$

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$B_{2,2} = \frac{1}{(1-2)(3-2)} = \boxed{-1}$$

Subs. $B_{0,2}$, $B_{1,2}$ and $B_{2,2}$ in I, where $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, We get

$$P_2(t) = 3\left[\frac{1}{2}\bar{e}^{t} + \frac{1}{2}\bar{e}^{3t} - \bar{e}^{2t}\right]$$

Similarly, We can deduce that

... ...

$$P_3(t) = 6\left[\frac{1}{8}\bar{e}^{t} + \frac{1}{4}\bar{e}^{3} - \frac{1}{3}\bar{e}^{2t} - \frac{1}{24}\bar{e}^{5t}\right]$$

* Solve it as a home Work.

Hint:

$$P_3(t) = \lambda_0\lambda_1\lambda_2\left[B_{0,3}\bar{e}^{\lambda_0 t} + B_{1,3}\bar{e}^{\lambda_1 t} + B_{2,3}\bar{e}^{\lambda_2 t} + B_{3,3}\bar{e}^{\lambda_3 t}\right]$$

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