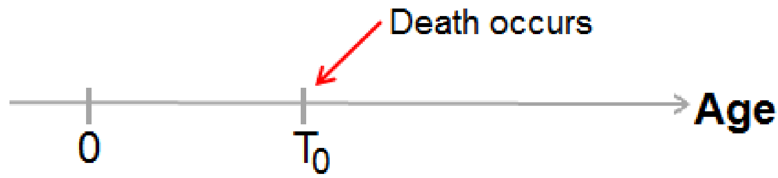
LECTURE 2

You are going to value cash flows they are paid at some time unknown future life

For example may insurance policy provide a benefit on the death of policy holder. In order to estimate the time at which a death benefit is payable. The insurer needs a model of human mortality.

From which probability of death practically can be calculated. Let us begin with the age of death random variable and denoted by T_0



The age at death random variable can take any value in $[0, \infty)$.

$$T_0 \in [0, \infty).$$

But sometime we assume that no individual can live beyond than certain any (highest age). we call that limity age and we denote it by ω

$$T_0 \in [0, \omega].$$

Suppose that T_0 a continuous random variable

$T_0 \in [0, \infty)$ => No limity age, $T_0 \in [0, \omega]$ =>limity age

We need probability distribution,

- $F_0(t) = \Pr(T_0 \leq t)$ the cumulative distribution function CDF of T_0
- $f_0(t) = \frac{d}{dt} F_0(t)$ the probability distribution function PDF of T_0

In life contingency we often need to calculate the probability that an individual will survive certain age.

We need to define the survival distribution function (SDF) of T_0:

$$S_0(t) = \Pr(T_0 > t) = 1 - \Pr(T_0 \leq t) = 1 - F_0(t)$$

$S_0(t)$ Represent the age-at-death random variable we can also call it future lifetime of a person age 0 now.

Not all function can be regarded as survival function. A survival function must satisfy the following:

1. $\boldsymbol{S_0(0) = 1}$
   That every individual can live at least 0 years
   $S_0(0) = \Pr(T_0 > 0) = 1 - \Pr(T_0 \le 0) = 1$

2. $\boldsymbol{S_0(\omega) = 0}$ **or** $\boldsymbol{\lim_{t\to\infty} S(t) = 0}$
   every individual must die eventually.

3. $\boldsymbol{S_0(t)}$ **is monotonically decreasing**
   $$\frac{d}{dt}S_0(t) = -\frac{d}{dt}F_0(t) = -f_0(t) < 0$$
   That means, the probability of surviving at age 80 can't be greater than the age 70.

SUMMARY:

- $f_0(t) = \frac{d}{dt}F_0(t) = -\frac{d}{dt}S_0(t)$
- $S_0(t) = \int_u^\infty f_0(u)du = 1 - \int_0^t f_0(u)du = 1 - F_0(t)$
- $\Pr(a < T_0 \le b) = \int_a^b f_0(u)du = F_0(b) - F_0(a) = 1 - S_0(b) - 1 + S_0(a) = S_0(a) - S_0(b)$
- If T_0 is continuous random variable, then $\Pr(T_0 = c) = 0, \forall\ any\ constant\ c$.

Example: Suppose that $S_0(t) = 1 - \frac{t}{100}, 0 \le t \le 100$

a) Verify that $S_0(t)$ is valid survival function.
b) Find expression for $F_0(t)$ and $f_0(t)$.
c) Calculate the probability that T_0 is greater than 30 and smaller than 60.

Solution:

a) $S_0(0) = 1 - \frac{0}{100} = 1 - 0 = 1$. ($1^{st}$ condition satisfies)
   $S_0(100) = 1 - \frac{100}{100} = 1 - 1 = 0$ (2^{nd} condition satisfies)
   The derivative of $S_0(t)$ is

$S'_0(t) = -\frac{1}{100} < 0$, decreasing ($3^{rd}$ condition satisfies)

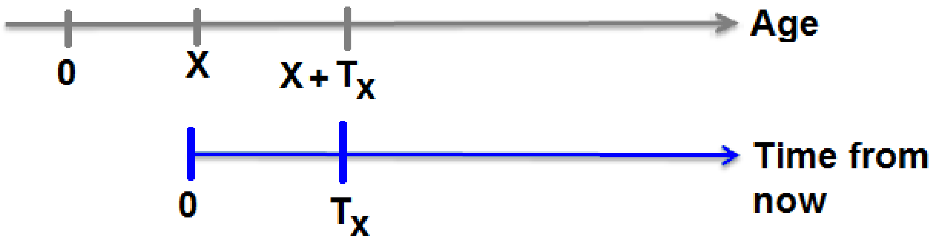
Then $S_0(t)$ is a valid function.

b) $F_0(t) = 1 - S_0(t) = \frac{t}{100}, 0 \le t \le 100$
   $f_0(t) = \frac{d}{dt}F_0(t) = \frac{1}{100}$
c) $\Pr(30 < t < 60) = S_0(30) - S_0(60) = 1 - \frac{30}{100} - 1 + \frac{60}{100} = 0.3$

1.2 Future lifetime random variable

Let us consider an individual whose age is X is now, and we donate by (x) such individual. We are interested on the future time of (x), we donate by:

T_X The future lifetime random variable for (x).



We call that, the line diagram.

If there's no limity age, then $T_X \in [0, \infty)$.

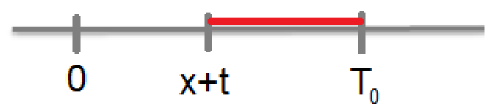
If the limit age is assumed, then $T_X \in [0, \omega - x)$.

$$T_X + x < \omega$$
$$T_X < \omega - x$$

(We need to subtract x because the individual has attains age x at time 0.)

Let $S_X(t)$ be the survival function for the future lifetime random variable, "x" is the age of individual is x now.

$$S_X(t) = \Pr(T_X > t)$$
$$= \Pr(T_0 > x + t \,/\, T_0 > x)$$
$$= \frac{\Pr((T_0 > x+t) \cap (T_0 > x))}{\Pr(T_0 > x)} \quad \text{(Conditional probability)}$$
$$= \frac{\Pr(T_0 > x+t)}{\Pr(T_0 > x)}$$
$$S_X(t) = \frac{S_0(x+t)}{S_0(x)}$$



$$x + t < T_0$$

Conditional probability:

$$\Pr(A \,/ B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}$$