PHYS 507
Lecture 4: Vector Integration

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Line Integrals-a

Using an increment of length $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$, we may encounter the line integrals:

$$\int_C \phi\, d\mathbf{r}, \quad \int_C \mathbf{V}\cdot d\mathbf{r}, \quad \int_C \mathbf{V}\times d\mathbf{r}$$

- In each of of which the integral is over some contour C that may open or closed.
- The first integral, with ϕ scalar, reduces immediately to

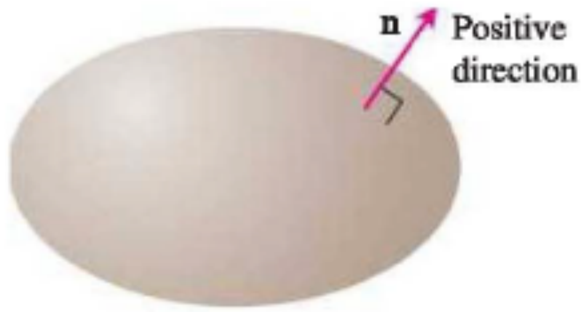
$$\int_C \phi\, d\mathbf{r} = \mathbf{i}\int_C \phi(x,y,z)\,dx + \mathbf{j}\int_C \phi(x,y,z)\,dy + \mathbf{k}\int_C \phi(x,y,z)\,dz$$

Line Integrals-b

- The previous relation is true only for Cartesian coordinates where the unit vectors are constant.
- The three integrals are Riemann integrals.
- The integral with respect to x cannot be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z.
- This means that the path of integration C must be specified, unless the integrand has some special properties that lead the integral to depend only on the value of the end points.

Surface Integrals-a

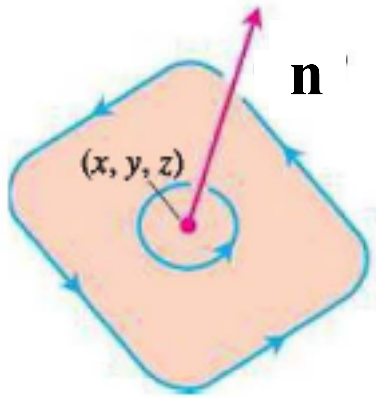
- Surface integrals appear in the same form as line integrals. The element of area is also a vector, $d\boldsymbol{\sigma}$. Often this area element is written $d\boldsymbol{\sigma} = \mathbf{n}dA$ in which $\mathbf{n}$ is the unit normal vector to indicate the positive direction.
- If the surface is closed, we agree to take the the outward normal as positive.



Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

Surface Integrals-b

- If the surface is open, the positive normal depends on the direction in which the perimeter of an open surface is traversed.
- If the right hand fingers are placed around the perimeter, the positive normal is indicated by the thumb of the right hand.



Surface Integrals-c

- Surface integrals appear in the forms:

$$\int \phi\mathbf{n}\,dA, \quad \int \mathbf{V}\cdot\mathbf{n}\,dA, \quad \int \mathbf{V}\times\mathbf{n}\,dA$$

The flux of **V** through the surface

Volume Integrals

Volume integrals are somewhat simpler, because the volume element $d\tau$ is a scalar quantity. We have:

$$\int_V \mathbf{V}\, d\boldsymbol{\tau} = \mathbf{i}\int_V V_x\left(x,y,z\right)d\boldsymbol{\tau} + \mathbf{j}\int_V V_y\left(x,y,z\right)d\boldsymbol{\tau} + \mathbf{k}\int_V V_z\left(x,y,z\right)d\boldsymbol{\tau}$$

Integral Definitions of differential Vector Operators

The following interesting relations relate the differential vector operators and the different forms of integrals we have seen so far:

$$\vec{\nabla}\phi = \lim_{\int d\tau \to 0} \frac{\int \phi \mathbf{n}\, dA}{\int d\tau}$$

$$\vec{\nabla} \cdot \mathbf{V} = \lim_{\int d\tau \to 0} \frac{\int \mathbf{V} \cdot \mathbf{n}\, dA}{\int d\tau}$$

$$\vec{\nabla} \times \mathbf{V} = \lim_{\int d\tau \to 0} \frac{\int \left(\mathbf{n} dA\right) \times \mathbf{V}}{\int d\tau}$$

The Gauss's Theorem-a

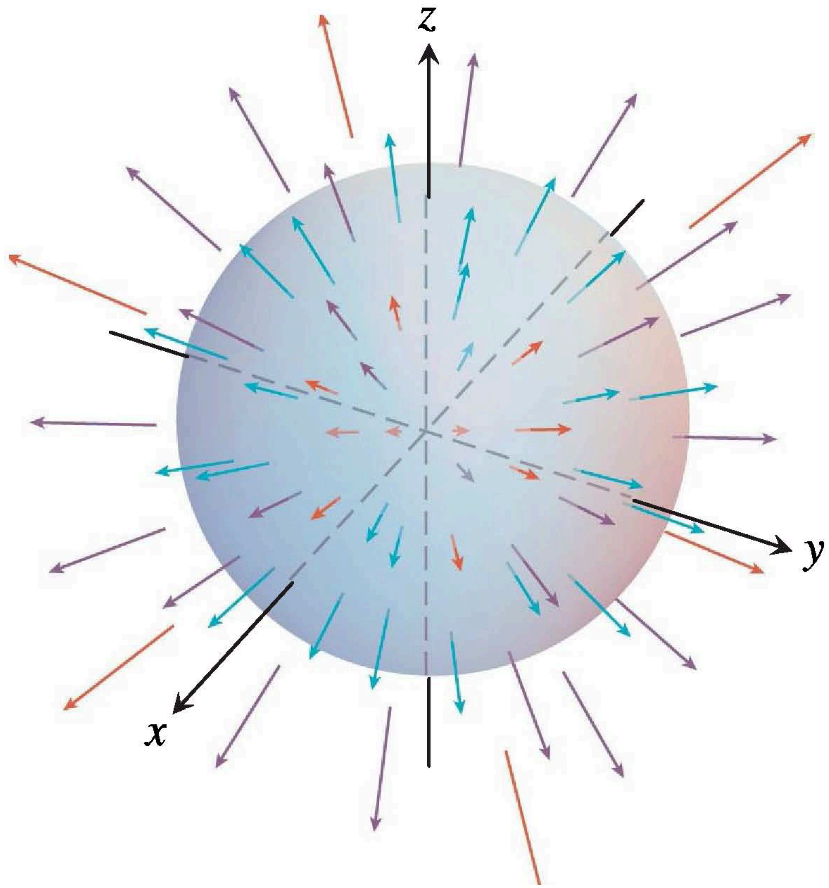
- Gauss's theorem relates the surface integral of a vector and the volume integral of the divergence of that vector provided that both the vector and its first partial derivatives are continuous over the region of interest.

$$\int_S \mathbf{V} \cdot \mathbf{n}\, dA = \int_V \vec{\nabla} \cdot \mathbf{V}\, d\boldsymbol{\tau}$$

- In words, the surface integral of a vector over a closed surface equals the volume integral of the divergence of the vector integrated over the volume enclosed by the surface.

The Gauss's Theorem-b

- From a physical point of view $\vec{\nabla}\cdot\mathbf{V}$ is the net outflow of fluid per unit volume. The volume integral then gives the total outflow.
- Two alternative form of Gauss's theorem are



$$\int_S V\mathbf{n}\,dA = \int_V \vec{\nabla}V\,d\tau$$

$$\int_S \mathbf{n}\,dA\times\mathbf{P} = \int_V \vec{\nabla}\times\mathbf{P}\,d\tau$$

The Green's Theorem

- A frequently useful corollary of Gauss's theorem is a relation known as Green's theorem. Consider two scalar functions u and v, then we have the following two alternate forms:

$$\int_V \left(u\vec{\nabla} \cdot \vec{\nabla} v - v\vec{\nabla} \cdot \vec{\nabla} u \right) d\boldsymbol{\tau} = \int_S \left(u\vec{\nabla} v - v\vec{\nabla} u \right) \cdot \mathbf{n}\, dA$$

$$\int_S u\vec{\nabla} v \cdot \mathbf{n}\, dA = \int_V u\vec{\nabla} \cdot \vec{\nabla} v\, d\boldsymbol{\tau} + \int_V \vec{\nabla} u \cdot \vec{\nabla} v\, d\boldsymbol{\tau}$$

The Stokes's Theorem-a

- Gauss's theorem relates the surface integral of a vector and the volume integral of the divergence of that vector over the closed surface bounding the volume. The Stokes's theorem provides us with an analogous relation between the surface integral of a derivative of a function and the line integral of the function, the path of integration being the perimeter bounding the surface.

$$\int_S \mathbf{V} \cdot \mathbf{n}\, dA = \int_V \vec{\nabla} \cdot \mathbf{V}\, d\boldsymbol{\tau}$$

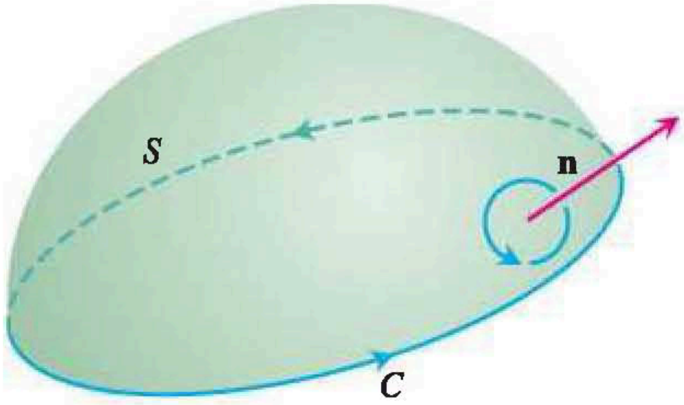
The Stokes's Theorem-b

- The form of the theorem is:

$$\oint \mathbf{V} \cdot d\mathbf{r} = \int_S \vec{\nabla} \times \mathbf{V} \cdot \mathbf{n}\, dA$$

- An alternate form is

$$\oint \phi d\mathbf{l} + \int_S \vec{\nabla}\phi \times \mathbf{n}\, dA = 0$$



The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$. If the thumb of a right hand points along $\mathbf{n}$, the fingers curl in the direction of C.