Lecture 4

complex manifolds, Kähler manifold

almost complex structure

Suppose dim $M = n$ is even, $n = 2m$. We may consider combining $2m$ coordinates into complex combinations,

$$z^i = x^{2i-1} + \sqrt{-1}x^{2i}, \quad (i = 1, ..., m).$$

But, what is a motivation for us to do this? More importantly, this would depend on a choice of coordinates. The manifold M must have some structure which compells us to introduce such complex combinations.

A natural structure would be a tensor field J^μ_ν ($\mu, \nu = 1, ..., n = 2m$), which has the property,

$$J^2 = -1.$$

(In today's lecture, I use Roman characters $i, j, ...$ for complex coordinates,and Greek characters $\mu, \nu, ...$ for real coordinates. So, $i, j... = 1, ..., n$ and $\mu, \nu, ... = 1, ..., m$.)

In a slightly more sophisticated way of saying, at each point $p \in M$, we have a linear map $J : T_pM \to T_pM$ obeying $J^2 = -1$. If there is such J, may consider diagonalizing it. However, since eigenvalues of J must be $\pm\sqrt{-1}$, we cannot do so in a vector space with real coefficients like T_pM. To do that, we need to allow vectors in T_pM have complex-valued coefficients. The complexified tangent space can be decompoesed into a holomorphic part T_pM^+ and an anti-holomorphic part T_pM^-, both m-dimensional over $\mathbf{C}$, and J has eigenvalues $\pm\sqrt{-1}$ on $T_pM^\pm$.

Note that it is not always possible to have such a tensor field J. For example, it is known that one cannot have such a tensor field on the 4-sphere S^4.

If we can define J on M satisfying $J^2 = -1$, we say that (M, J) is an *almost* complex manifold. The tensor J is called almost complex structure.

We say *almost* since we do not yet have a fully complex manifold. To be called a complex manifold, on each coordinate patch U, we need to be able to define complex coordinates z^i ($i = 1, ..., m$) so that $\{\partial/\partial z^i\}$ gives basis for the holomorphic part T_pM^+ of the tangent space at each point p on U. In order for this to be possible, the tensor field J has to satisfy the following differential equations,

$$J^\nu_\mu \partial_\rho J^\mu_\sigma - J^\nu_\mu \partial_\sigma J^\mu_\rho - J^\mu_\sigma \partial_\mu J^\nu_\rho + J^\mu_\sigma \partial_\rho J^\nu_\mu = 0.$$

The left-hand side of this equation is known as the Nijenhuis tensor.

If the Nijenhuis tensor vanishes, we can find holomorphic coordinates z^i so that $J(\partial/\partial z^i) = \sqrt{-1}\partial/\partial z^i$. Since J is defined globally on M, it follows that the manifold can be covered smoothly with complex coordinate charts. In holomorphic coordinates z^i, the tensor J is of the form,

$$J^i_j = \sqrt{-1}\delta^i_j, \quad J^{\bar{i}}_{\bar{j}} = -\sqrt{-1}\delta^{\bar{i}}_{\bar{j}},$$
$$J^i_{\bar{j}} = 0, \quad J^{\bar{i}}_j = 0.$$

Question 1: Show that, when two coordinatepatches U and V overlap, the transition function between the two complex coordinates obeying the above set of conditions are holomorphic.

Not all almost complex manifolds can be made to complex manifolds. For example, we can define an almost complex structure on the 6-sphere S^6, but its Nijenhuis tensor does not vanish.

Kähler manifolds

So far, we did not assume that M is equipped with a metric $g_{\mu\nu}$. When it exists, it is natural to impose the following compatibility conditions,

$$g_{\mu\nu} J^\mu_\rho J^\nu_\sigma = g_{\rho\sigma}, \quad \nabla_\mu J^\nu_\rho = 0.$$

These imply that the Nijenhuise tensor vanishes and therefore (M, J) is a complex manifold. But, it implies more. The first condition implies that, in complex coordinates z^i, the only non-zero components of the metric is $g_{i\bar{j}}$. Namely,

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = 2i g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}.$$

To understand what the second equation means, we introduce the Kähler form,

$$k = \frac{1}{2} g_{\mu\nu} J^\mu_\rho dx^\rho \wedge dx^\nu = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

The second condition then implies that k is closed, $dk = 0$. In components, it means

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}, \quad \partial_{\bar{j}} g_{i\bar{k}} = \partial_{\bar{k}} g_{i\bar{j}}.$$

Locally, we can always integrate these equations as,

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z, \bar{z}),$$

for some function $K(z, \bar{z})$, which is known as the Kähler potential. It is unique up to Kähler transformation, $K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})$ for any holomorphic function $f(z)$. We should note that the Kähler potential cannot be a globally defined smooth function on a compact Riemannian manifold M without boundary. This is because $J^m/m!$ is equal to the volume form vol (dim $M = 2m$). If K were globally defined, J would be an exact form and so is vol $= J^m/m!$. If the volume form were exact, its integral over M would vanish, which would be inconsistent with the assumption that the metric is non-degenerate.

Given the metric in the above form, we can compute the affine connection (Christoffel symbol).

Question 2: Show that the only non-zero components are

$$\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}},$$

and its complex conjugate. Components with mixed indices (mixed in i and $\bar{j}$) all vanish.

Thus, the only non-zero components of the curvature tensor are,

$$R_{\bar{i}j}{}^k{}_l = \partial_{\bar{i}} \Gamma^k_{jl}.$$

The connection and the curvature describe how tangent vectors are parallel transported on the manifold M, in such a way that the metric structure is respected, $\nabla_\mu g_{\nu\rho} = 0$. For a general

Riemannian manifold without complex structure, the anti-symmetry $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$ means that the transport is done by an element of the orthogonal group $SO(n)$ (n =dim$M$). For the Kähler manifold, we can use complex coordinates z^i, and the curvature takes the form $R_{\bar{i}j}{}^k{}_l$. This means that the transport is done by an element of the unitary group $U(m) \in SO(n = 2m)$.

When we start at $p \in M$, pick a tangent vector, and transport a tangent vector by using the affine connection. We move around the manifold M and come back to the same point p. We may not get the same vector as the one we started with. On a general Riemannian manifold, the vectors before and after are related by an element of $SO(n)$. For each closed path on M, we can associate an element of $SO(n)$. It is called a holonomy for the path. If we consider holonomies of all paths starting and ending at $p \in M$, it makes a group – the holonomy group. We could start with another point on M. As far as M is connected, the resulting holonomy group are equivalent. When M is Kähler, the holonomy group is a subgroup of $U(m) \in SO(2m)$.

Question 3: Show that the Ricci tensor for a Kähler manifold takes a particularly simple form,

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \text{logdet}(g).$$

This expression will be useful when we discuss Calabi-Yau manifolds later.

Hodge-de Rham cohomology

On a Kähler manifold, we can consider forms that contains p dz's and q $d\bar{z}$'s. The space of k-forms $C^k(M)$ can then be decomposed into,

$$C^k(M) = \oplus_{p+q=k} C^{p,q}(M),$$

where elements of $\omega \in C^{p,q}(M)$, called (p,q)-forms, are of the form,

$$\omega = \frac{1}{p!q!} \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_q}.$$

Similarly, the exterior derivative and its adjoint $\delta = - * d*$ (note: the sign factor is (-1) since M is even dimensional) can be split as

$$d = \partial + \bar{\partial}, \quad \delta = \partial^\dagger + \bar{\partial}^\dagger,$$

where

$$\partial \omega = dz^i \wedge \left(\frac{\partial}{\partial z^i} \omega \right), \quad \bar{\partial} \omega = d\bar{z}^{\bar{j}} \wedge \left(\frac{\partial}{\partial \bar{z}^{\bar{j}}} \omega \right),$$

and

$$\partial^\dagger = - * \partial *, \quad \bar{\partial}^\dagger = - * \bar{\partial} * .$$

These operators $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger$ map (p,q)-forms into $(p+1,q)$, $(p,q+1)$, $(p-1,q)$, and $(p,q-1)$-forms, respectively.

Question 4: Show that the Laplace-Beltrami operator is also decomposed as

$$\Delta = d\delta + \delta d = 2(\partial \partial^\dagger + \partial^\dagger \partial) = 2(\bar{\partial} \bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial}),$$

and

$$\partial \bar{\partial}^\dagger + \bar{\partial}^\dagger \partial = 0, \quad \bar{\partial} \partial^\dagger + \partial^\dagger \bar{\partial} = 0.$$

We can consider cohomology of (p,q)-forms with respect to $d = \partial + \bar{\partial}$. It is called the Hodge-de Rham cohomology and denoted by $H^{p,q}(M)$. Since ∂ and $\bar{\partial}$ change degrees of forms differently, elements of $H^{p,q}(M)$ are annihilated by both ∂ and $\bar{\partial}$, modulo images of ∂ and $\bar{\partial}$. This gives the decomposition,

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

Since the de Rham cohomology $H^k(M)$ is generated by harmonic forms, so is each $H^{p,q}(M)$ in the above decomposition.

On a compact Kähler manifold without boundary, the Kähler form $k = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ always generate a non-trivial element of $H^{1,1}(M)$ since J^m is proportional to the volume form (dim $M = 2m$). Thus, $\dim H^{1,1}(M) \geq 1$. The $(1,1)$-cohomology defined by k is called the Kähler class.

Another potential generator of $H^{1,1}(M)$ is the Ricci form, $R_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$. Later in this course, we will learn that it is related to the first Chern class of M. On a Calabi-Yau manifold, the Ricci form is zero. It was conjectured by Calabi that, if the Ricci form is an exact form (i.e., trivial as an element of $H^{1,1}(M)$), we can always choose a metric on M with the same complex structure and the Kähler class such that the Ricci curvautre is identically equal to zero. This conjecture was proven by Yau, and a Kähler manifold with zero Ricci curvature is called a Calabi-Yau manifold.

hyper-Kähler manifolds

Suppose there are more than one J's that satisfy the compatibility conditions with the metric,

$$g_{\mu\nu} J_\rho^{(a)\mu} J_\sigma^{(a)\nu} = g_{\rho\sigma}, \quad \nabla_\mu J_\rho^{(a)\nu} = 0, \quad (a = 1, ..., N-1).$$

The complex structure we considered in the above corresponds to the case of $N = 2$. (The reason we set the number of J's to be $(N-1)$ will become clear later.) Since $J^{(a)}$ is covariantly constant, it is invariant under parallel transport. Therefore, when we start at a point $p \in M$, transport $J^{(a)}$ along a closed path, and come back to the same p, then $J^{(a)}$ does not change. This means that $J^{(a)}$ commutes with the holonomy group of M. By Shur's lemman in group theory, one can show that $J^{(a)}$ must for a division algebra over real numbers. Namely, they make either the complex numbers, in which case $N = 2$ and there is a single complex structure J, or the quaternions, in which case $N = 2$ and thre are three possible imaginary units $J^{(1)}, J^{(2)}, J^{(3)}$. As we saw in the above, the holonomy group for $N = 2$ case is a subgroup of $U(m)$. The second case with $N = 4$ requires that dim $M = 4r$ and the holonomy group is a subgroup of $Sp(r) \in U(2r) \in SO(4r)$. In this case, M is called hyper-Kähler. The Ricci curvature of a hyper-Kähler manifold is zero.

There are manifolds with other types of holonomies. For simply connected Riemannian manifolds which are not locally a product space and are not a symmetric space (space with continous group symmetry preserving its metric), possible holonomy groups are:

$SO(n)$, $U(n)$, $SU(n)$, $Sp(n) \cdot Sp(1)$, $Sp(n)$, G_2, and Spin (7). A manifold with $SU(n)$ holonomy is a Calabi-Yau manifold, and a manifold with $Sp(n) \cdot Sp(1)$ holonomy is known as a quaternionic-Kähler manifold.

extended supersymmetry

On a Kähler manifold, differential forms are generated by dz^i and $d\bar{z}^{\bar{i}}$. Correspondingly, we can consider fermions,

$$\bar{\psi}^i \leftrightarrow dz^i\wedge, \quad \bar{\psi}^{\bar{i}} \leftrightarrow d\bar{z}^{\bar{i}}\wedge,$$

and

$$\psi^i \leftrightarrow (-1)^{nk+k+1} * dz^i *, \quad \psi^{\bar{i}} \leftrightarrow (-1)^{nk+k+1} * d\bar{z}^{\bar{i}} * .$$

They obey the anti-commutation relations,

$$\{\psi^i, \bar{\psi}^{\bar{j}}\} = \{\bar{\psi}^i, \bar{\psi}^{\bar{j}}\} = g^{i\bar{j}}, \quad \{(\text{others})\} = 0.$$

If we identify p_i and $p_{\bar{j}}$ by $-i\partial/\partial z^i$ and $-i\partial/\partial \bar{z}^{\bar{j}}$ according to quantum mechanics, we can write

$$\partial = i\bar{\psi}^i p_i, \quad \bar{\partial} = i\bar{\psi}^{\bar{j}} p_{\bar{j}}.$$

These, together with their hermitian conjugates, generate extended supersymmetry. We have twice as many supersymmetry generators (supercharges), so the resulting symmetry is called $N = 2$ supersymmetry.

Similarly, on a hyper-Kähler manifold, we have 4 times more supercharges generating $N = 4$ supersymmetry.