Some Definitions for Bivariate r.v (X, Y)

Defn ①: Covariance

التغاير

$$Cov(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E(XY) - \mu_X \mu_Y$$

التغاير تقيس درجة التوافق بين المتغيرين X, Y

Defn ②: Correlation Coefficient

معامل الارتباط

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \, \sigma_Y}$$

σ_X: Stand. dev. of X

σ_Y: stand. dev. of Y

where $-1 \le \rho \le 1$.

Some props

① For independent r.v's X and Y, $Cov(X,Y) = 0$ and Consequently $\rho(X,Y) = 0$

i.e indep. r.v's $\Rightarrow \rho(X,Y) = 0$

Note that For $\rho(X,Y) = 0$, It's not necessary for X and Y to be independent r.v's

② $Cov(X,X) = Var(X) = \sigma_X^2$

③ $Cov(\alpha X, \beta Y) = \alpha\beta \, Cov(X,Y)$ where $\alpha, \beta \in \mathbb{R}$

④ $Var(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2Cov(X,Y)$

Not that: For $Z = aX + bY$, $Var(aX+bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,Cov(X,Y)$

OR $Var(Z) = a^2\sigma_X^2 + 2ab\,\rho\,\sigma_X\sigma_Y + b^2\sigma_Y^2$ See P.32 Textbook

1 Lecture (7) **Normal distribution**

The Normal dist^{n} $X \sim N(\mu, \sigma^2)$ — μ: mean, σ^2: variance

its pdf is
$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

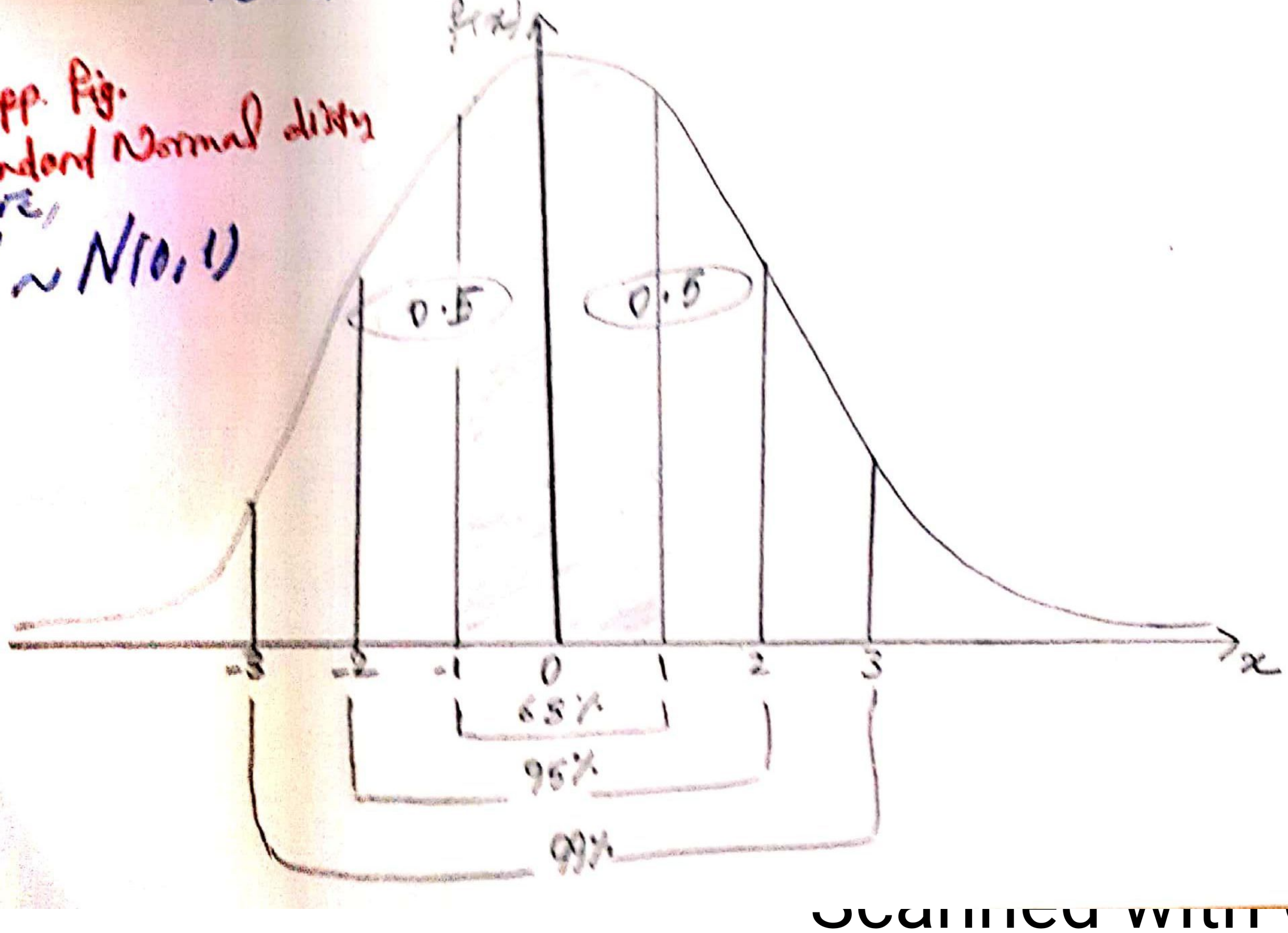
For $X \sim N(0,1)$ (μ, σ) which is known as standard Normal dist^{n}, its pdf is
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

and its prob. dist^{n} cdf is
$$\Phi(x) = \int_{-\infty}^{x} f(\xi)\, d\xi$$

ξ آكساي

قيم دالة التوزيع الطبيعي هذه توجد في جداول خاصة.

In opp. fig. Standard Normal dist^{n} where, $X \sim N(0,1)$



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- **The Joint Normal distn** See p.31
(Bivariate Normal distn)

Its p.d.f is

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \cdot \exp\left[-\frac{1}{2} Q(x,y)\right]$$

$$, \quad -\infty < x,y < \infty$$

where

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

$, -1 < \rho < 1.$

- **Bivariate Normal distn theorem**

If X and $Y \sim$ Bivariate Normal distn then

$$Z = aX + bY \sim \text{Normal distn}$$

with mean $E(Z) = a\mu_X + b\mu_Y$

and $Var(Z) = a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2$

or $Var(Z) = a^2\sigma_X^2 + 2ab\,Cov(X,Y) + b^2\sigma_Y^2$

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pb 1.4.5 p.32

Let X and Y have the joint normal distribution described before, what value of α minimizes the variance of $Z = \alpha X + (1-\alpha) Y$? Simplify your result when X and Y are independent.

Ans: $Z = \alpha X + (1-\alpha) Y$

$$Var(Z) = \alpha^2 \sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$$

$$Var(Z) = \alpha^2\sigma_X^2 + (2\alpha - 2\alpha^2)\rho\sigma_X\sigma_Y + (1-2\alpha+\alpha^2)\sigma_Y^2$$

To minimize $V = Var(Z)$ by using α

Diff. V w.r.t α and then equating by zero, i.e let $\frac{\partial V}{\partial \alpha} = 0$ as follows.

$$2\alpha\sigma_X^2 + (2-4\alpha)\rho\sigma_X\sigma_Y + (-2+2\alpha)\sigma_Y^2 = 0$$

$$\therefore 2\alpha\sigma_X^2 - 4\alpha\rho\sigma_X\sigma_Y + 2\alpha\sigma_Y^2 = 2\sigma_Y^2 - 2\rho\sigma_X\sigma_Y$$

$$\therefore \cancel{2}\alpha(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2) = \cancel{2}(\sigma_Y^2 - \rho\sigma_X\sigma_Y)$$

$$\therefore \alpha = \alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}, \quad -1 \le \rho < 1$$

- For independent r.v.s X and Y, $\rho = 0$, in this case

$$\alpha^* = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$$

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