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Ch 2: Conditional probability and Conditional Expectation

For Discrete Bivariate r.v (X, Y), we have

Defn

$$P_{X|Y}(x|y) = \frac{pr(X=x \text{ and } Y=y)}{pr(Y=y)}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Joint prob. mass fn of (X,Y)

Conditional prob. mass fn of X given $Y=y$

marginal mass fn of Y

$$pr(A|B) = \frac{pr(A\cap B)}{pr(B)}, \quad pr(B)\neq 0$$

$x, y = 0, 1, 2, \cdots,$

$P_Y(y) \neq 0$

$$\therefore \quad P_{X,Y}(x,y) = P_{X|Y}(x|y)\, P_Y(y) \qquad (1)$$

$$\& \because \quad pr(X=x) = \sum_y P_{X,Y}(x,y) \qquad (2)$$

(1), (2) $\Rightarrow$

$$\therefore \quad pr(X=x) = \sum_{y=0}^{\infty} P_{X|Y}(x|y)\, P_Y(y)$$

which is the marginal mass fn of X, this formula is called the law of total probability.

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EX: Suppose X has a binomial distribution with parameter p and N, where N has a poisson distribution with mean λ. what is the marginal distn for X?

Ans:
$$X \sim Bin(p, N), \quad N \sim poisson(\lambda)$$

$$pr(X = x) \;??$$

$$P_{X|N}(x|n) = \binom{n}{x} p^{x} (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$

and
$$P_N(n) = \frac{e^{-\lambda}\lambda^{n}}{n!}, \quad n = 0, 1, 2, \dots$$

$$\therefore pr(X = x) = \sum_{n=0}^{\infty} P_{X|N}(x|n)\, P_N(n)$$

$$= \sum_{n=0}^{\infty} \binom{n}{x} p^{x}(1-p)^{n-x} \cdot \frac{e^{-\lambda}\lambda^{n}}{n!}$$

$$\therefore pr(X=x) = \lambda^{x} p^{x} e^{-\lambda} \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!\,(n-x)!} \cdot \frac{(1-p)^{n-x}\lambda^{n-x}}{\cancel{n!}}$$

$$\therefore pr(X=x) = \frac{(\lambda p)^{x} e^{-\lambda}}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!}$$

let $n - x = r$

$$\therefore pr(X=x) = \frac{(\lambda p)^{x} e^{-\lambda}}{x!} \sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^{r}}{r!}$$

$$\therefore pr(X=x) = \frac{(\lambda p)^x e^{-\lambda} e^{\lambda(1-p)}}{x!}$$

$$\therefore pr(X=x) = \frac{(\lambda p)^x e^{-\lambda p}}{x!} , \quad x = 0,1,2,\ldots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$\therefore X \sim poisson(\lambda p)$ with mean and Variance λp #

Defn For Continuous Bivariat r.v (X,Y), we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

joint prob. density fn of (X,Y)

marginal density fn of Y

Conditional prob. density fn of X given $Y=y$

$$, \quad x,y > 0, \quad f_Y(y) \neq 0$$

where $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$