QUANTUM MECHANICS: LECTURE 9

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Abstract

This lecture discusses the quantum harmonic oscillator by the Ladder operator method

CONTENTS

1 QUANTIZATION OF THE SHO HAMILTONIAN

From lecture (1) we have the classical Hamiltonian for the simple harmonic oscillator (SHO) :

$$H(p,x) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

Using the postulates of quantum mechanics discussed before, we obtain- upon quantization - the Hamiltonian operator :

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \tag{2}$$

The Hilbert space of which $\hat{X}$ and $\hat{P}$ act on is

$$\mathcal{L}^2(-\infty, +\infty; dx)$$

We now introduce the dimensionless Hamiltonian :

$$\hat{H}' = \frac{1}{2m\hbar\omega}\hat{P}^2 + \frac{1}{2}\frac{m\omega}{\hbar}\hat{X}^2 \tag{3}$$

This operator can be factorised and written in terms of 'creation' and 'inhalation 'operators; $a^\dagger$ and a respectively :

$$\hat{H}' = a^\dagger a + \frac{1}{2}I \tag{4}$$

with:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P} \tag{5a}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P} \tag{5b}$$

These operators, along with $\hat{H}'$, satisfy a well-known commutation relations.

The operators a, $a^\dagger$ and H' along with the commutator operation $[\cdot,\cdot]$ satisfy the $su(1,1)$ algebra.

$$[a, a^\dagger] = I \tag{6a}$$

$$[a, H'] = a \tag{6b}$$

$$[a^\dagger, H'] = -a^\dagger \tag{6c}$$

We also define the **number operator** $N \equiv a^\dagger a$ that acts on the eigenstates $|n\rangle$ resulting an eigenvalue of n :

$$N|n\rangle = n|n\rangle$$

as a result we may conclude that :

$$a|0\rangle = 0 \tag{7}$$

acting on the 'ground state'by the inhalation operator, kills it . Moreover :

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{8}$$

$$a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle \tag{9}$$

Hence, The Hamiltonian acting on these states will result (the energy eigenvalue) :

$$\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle \tag{10}$$

Implying that the 'number states' are the excitation states for the quantum harmonic oscillator. The creation and inhalation operators excite or deceit it, and it has a descrete energy spectrum of :

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \tag{11}$$

Even in the ground state, the quantum harmonic oscillator has a non-vanishing energy. This is a direct result for the uncertainty principle in time and energy.

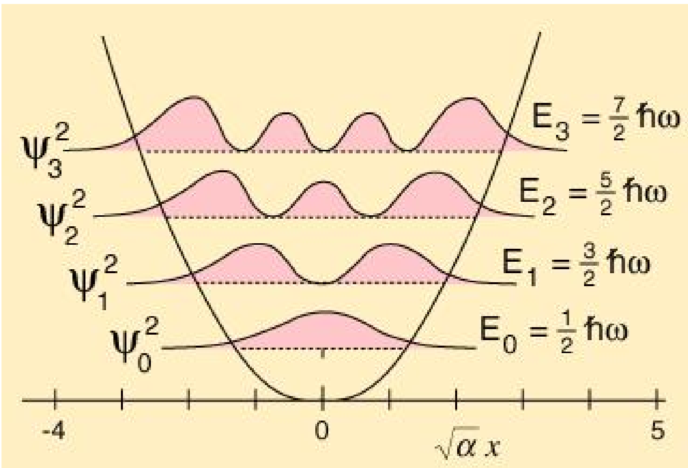

Figure 1: Energy-levels and wavefunctions of the quantum harmonic oscillator

2 THE EIGENFUNCTIONS

Since we introduced the eigenstates for the Hamiltonian (or the number operator equivalently). $|n\rangle$. We can use the ladder operator method to solve Scrödinger's equation in the position representation and find $\psi_n(x) = \langle x|n\rangle$. we start from (7):

$$a\psi_0(x) = 0 \tag{12}$$

resulting the differential equation :

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x) = 0 \tag{13}$$

having the solution:

$$\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2} \tag{14}$$

We can find the normalisation factor by:

$$\int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}x^2} dx = \frac{1}{|A|^2}. \quad (15)$$

Which is a typical Gaussian, hence A is found to be:

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \quad (16)$$

Now, in order to find the nth wavefunction $\psi_n(x)$, we first need to prove the following identity:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \quad (17)$$

Proof:

$$\frac{a^\dagger}{\sqrt{n}} \mid n-1\rangle = \frac{(a^\dagger)^2}{\sqrt{n(n-1)}} \mid n-2\rangle = \cdots = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

Thereby,

$$\psi_n(x) = \frac{(a^\dagger)^n}{\sqrt{n!}}\psi_0 x \quad (18)$$

Writing the expression explicitly , we obtain :

$$\psi_n(x) \equiv \langle x \mid n\rangle = \frac{1}{\sqrt{2^n n!}}\ \pi^{-1/4}\exp(-x^2/2)H_n(x) \quad (19)$$

Where $H_n(x)$ is the nth Hermite polynomial, that having the generating formula (Rodrigues's formula)

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad (20)$$

They are one of the classical orthogonal polynomials.
Note that this result can be obtained by solving immediately the Schrödinger's equation (using serier solution, or Sturm-Liouville theorem).

3 COHERENT STATES

Coherent states are a very important topic in quantum mechanics. Coherent states are quantum states that display an oscillatory behaviour similar to the one displayed in the simple harmonic oscillator. Formally, a coherent state is a state that has a minimum uncertainty, and takes the form (where a here is the inhalation operator) :

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad (21)$$

The ground state $|0\rangle$ is a coherent state. Since it has the minimum ucertainty :

$$\langle\Delta X\rangle_0\langle\Delta P\rangle_0 = \frac{\hbar}{2} \quad (22)$$

This is not however the case for the stationary states $|n\rangle$, as we can show that:

$$\langle\Delta X\rangle_n\langle\Delta P\rangle_n = \frac{\hbar}{2}(2n+1) \quad (23)$$

Hence the states $\alpha\rangle$, the coherent states of the Harmonic oscillator are not stationary states. Their time evlution is important in the classical limit, as it leads to Ernfest theorem, and one can obtain from them the Classical equations of motion for the Harmonic oscillator. Hence the classical harminic oscilator is the limit for the coherent states $|\alpha\rangle$, not the states $|n\rangle$. However the detailed mathematical argument is beyond the scope of this course, the interested reader might want to refer to any Textbook in the references for details.

REFERENCES