Integral Calculus

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Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if

$$F^{'}(x) = f(x) \text{ for every } x \in I.$$

Antiderivatives

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Example

1. Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.
   Since $F^{'}(x) = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.
2. Let $G(x) = \sin\ x + x$ and $g(x) = \cos\ x + 1$.
   Since $G^{'}(x) = \cos\ x + 1$, then the function $G(x)$ is an antiderivative of $g(x)$.

Theorem

If functions F and G are antiderivatives of f on an interval I, there exists a constant c such that $G(x) = F(x) + c$ for every $x \in I$.

Example

Let $f(x) = 2x$. The functions
$F(x) = x^2 + 2$,
$G(x) = x^2 - \frac{1}{2}$,
$H(x) = x^2 - \sqrt[3]{2}$,
are antiderivatives of the function f. Therefore, $F(x) = x^2 + c$ is the general form of the antiderivatives (the family) of the function $f(x) = 2x$.

Example

Find the general form of the antiderivatives of $f(x) = 6x^5$.

Solution:
The function $F(x) = x^6 + c$ is the general antiderivative of f because $F'(x) = 6x^5$.

Indefinite Integrals

Definition

Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x)\ dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in the previous example are

1. $\int (2x+3)\ dx = x^2 + 3x + c.$
2. $\int (\cos\ x + 1)\ dx = \sin\ x + x + c.$

Basic Integration Rules

Rule 1: Power of x.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \quad \text{so} \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1$$

In words, to integrate the function x^n, we add 1 to the power (i.e., $n+1$) and divide the function by $n+1$. If $n = 1$, we have a special case

$$\int 1 \, dx = x + c.$$

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Rule 2: Trigonometric functions.

$$\frac{d}{dx}\sin\ x = \cos\ x, \quad \text{so} \quad \int \cos\ x \, dx = \sin\ x + c$$

$$\frac{d}{dx}\cos\ x = -\sin\ x, \quad \text{so} \quad \int -\sin\ x \, dx = \cos\ x + c$$

Therefore, $\int \sin\ x \, dx = -\cos\ x + c$.

The other trigonometric functions with the previous rules are listed in the following table:

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1 \ dx = x + c$
$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n, \ n \neq -1$	$\int x^n \ dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin \ x) = \cos \ x$	$\int \cos \ x \ dx = \sin \ x + c$
$\frac{d}{dx}(-\cos \ x) = \sin \ x$	$\int \sin \ x \ dx = -\cos \ x + c$
$\frac{d}{dx}(\tan \ x) = \sec^2 \ x$	$\int \sec^2 \ x \ dx = \tan \ x + c$
$\frac{d}{dx}(-\cot \ x) = \csc^2 \ x$	$\int \csc^2 \ x \ dx = -\cot \ x + c$
$\frac{d}{dx}(\sec \ x) = \sec \ x \ \tan \ x$	$\int \sec \ x \ \tan \ x \ dx = \sec \ x + c$
$\frac{d}{dx}(-\csc \ x) = \csc \ x \ \cot \ x$	$\int \csc \ x \ \cot \ x \ dx = -\csc \ x + c$

Example

Evaluate the integral.

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Solution:

1. $\int x^{-3}\, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$

Example

Evaluate the integral.

1. $\displaystyle\int x^{-3}\, dx$
2. $\displaystyle\int \frac{1}{\cos^2\ x}\, dx$

Solution:

1. $\displaystyle\int x^{-3}\, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$
2. $\displaystyle\int \frac{1}{\cos^2\ x}\, dx = \int \sec^2\ x\, dx = \tan\ x + c.$

   $\left(\sec\ x = \frac{1}{\cos\ x} \Rightarrow \sec^2\ x = \frac{1}{\cos^2\ x}\right)$

Properties of the Indefinite Integral

Theorem

Assume f and g have antiderivatives on an interval I, then

1. $\dfrac{d}{dx}\displaystyle\int f(x)\ dx = f(x).$
2. $\displaystyle\int \frac{d}{dx}(F(x))\ dx = F(x) + c.$
3. $\displaystyle\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$
4. $\displaystyle\int kf(x)\ dx = k\int f(x)\ dx,$ *where k is a constant.*

Example

Evaluate the integral.

1. $\displaystyle\int (4x+3)\ dx$
2. $\displaystyle\int (2\sin\ x + 3\cos\ x)\ dx$
3. $\displaystyle\int (\sqrt{x} + \sec^2\ x)\ dx$
4. $\displaystyle\int \frac{d}{dx}(\sin\ x)\ dx$
5. $\dfrac{d}{dx}\displaystyle\int \sqrt{x+1}\ dx$

Solution:

1. $\int (4x+3)\ dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$
2. $\int (2\sin\ x + 3\cos\ x)\ dx = -2\cos\ x + 3\sin\ x + c.$
3. $\int (\sqrt{x} + \sec^2\ x)\ dx = \frac{x^{\frac{3}{2}}}{3/2} + \tan\ x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan\ x + c.$
4. $\int \frac{d}{dx}(\sin\ x)\ dx = \sin\ x + c.$
5. $\frac{d}{dx}\int \sqrt{x+1}\ dx = \sqrt{x+1}.$

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5. $\displaystyle\frac{d}{dx}\int \sqrt{x+1}\ dx = \sqrt{x+1}.$

Example

If $\displaystyle\int f(x)\ dx = x^2 + c_1$ and $\displaystyle\int g(x)\ dx = \tan\ x + c_2$, find $\displaystyle\int \big(3f(x) - 2g(x)\big)\ dx.$

Solution:

1. $\int (4x+3)\ dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$
2. $\int (2\sin\ x + 3\cos\ x)\ dx = -2\cos\ x + 3\sin\ x + c.$
3. $\int (\sqrt{x} + \sec^2\ x)\ dx = \frac{x^{\frac{3}{2}}}{3/2} + \tan\ x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan\ x + c.$
4. $\int \frac{d}{dx}(\sin\ x)\ dx = \sin\ x + c.$
5. $\frac{d}{dx}\int \sqrt{x+1}\ dx = \sqrt{x+1}.$

Example

If $\int f(x)\ dx = x^2 + c_1$ and $\int g(x)\ dx = \tan\ x + c_2$, find $\int (3f(x) - 2g(x))\ dx$.

Solution:

From the third and fourth properties,
$\int (3f(x) - 2g(x))\ dx = 3\int f(x)\ dx - 2\int g(x)\ dx = 3x^2 - 2\tan\ x + c$, where $c = 3c_1 - 2c_2$.

Example

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Solution:

$$\int f'(x)\,dx = \int x^3\,dx$$
$$\Rightarrow f(x) = \frac{1}{4}x^4 + c.$$

If $x = 0$, then $f(0) = \frac{1}{4}(0)^4 + c = 1$ and this implies $c = 1$. Hence, the solution of the differential equation is $f(x) = \frac{1}{4}x^4 + 1$.

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Notes:

- When evaluating integrals, we can always check our answers by deriving the results.
- In the previous examples, we use x as a variable of the integration. However, for this role, we can use any variable such as y, z, t, etc . That is, instead of $f(x)\, dx$, we can integrate $f(y)\, dy$ or $f(t)\, dt$.
- The properties of the indefinite integral and the basic integral rules given in Table **??** are elementary for simple functions. For more complex functions, we need some techniques to simplify the integrals. Section **??**, we shall provide one of these techniques.

Theorem

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g. If F is an antiderivative of the function f on J, then

$$\int f(g(x))g^{'}(x)\ dx = F(g(x)) + c, \quad x \in I.$$

Example

Evaluate the integral $\int 2x\ (x^2+1)^3\ dx$.

Integration By Substitution

Theorem

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g. If F is an antiderivative of the function f on J, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c, \quad x \in I.$$

Example

Evaluate the integral $\int 2x\,(x^2+1)^3\,dx$.

Solution:

We can use the previous theorem as follows:
let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $f\big(g(x)\big) = (x^2+1)^3$. Since $g'(x) = 2x$, then from the theorem, we have

$$\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + c.$$

We can end with the same solution by using the five steps of the substitution method given below.

Steps of the integration by substitution:

Step 1: Choose a new variable u.

Step 2: Determine the value of du.

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral in terms of u.

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x.

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Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x.

In the previous example, let $u = x^2 + 1$, then $du = 2x\ dx$. By substituting that into the original integral, we have

$$\int u^3\ du = \frac{u^4}{4} + c.$$

Now, by returning the evaluation to the initial variable x, we have
$\int 2x(x^2+1)^3\ dx = \frac{(x^2+1)^4}{4} + c.$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$.

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Solution:

We use the theorem for the integral $2 \int \frac{\sec^2 \sqrt{x}}{2\sqrt{x}} \, dx$. Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $f(g(x)) = \sec^2 \sqrt{x}$. Since $g'(x) = 1/(2\sqrt{x})$, then we have

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$.

Solution:

We use the theorem for the integral $2\int \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}\, dx$. Let $f(x) = \sec^2\ x$ and $g(x) = \sqrt{x}$, then $f(g(x)) = \sec^2\ \sqrt{x}$. Since $g'(x) = 1/(2\sqrt{x})$, then we have

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx = 2\tan\ \sqrt{x} + c.$$

By using the steps of the substitution method, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\, dx$. By substitution, we obtain

$$2\int \sec^2\ u\ du = 2\tan\ u + c = 2\tan\ \sqrt{x} + c.$$

Example

Evaluate the integral $\displaystyle\int \frac{x^2-1}{(x^3-3x+1)^6}\ dx$.

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Solution:

Let $u = x^3 - 3x + 1$, then $du = 3(x^2-1)\, dx$. By substitution, we have

$$\frac{1}{3}\int u^{-6}\, du = \frac{1}{3}\,\frac{1}{-5u^5} + c = \frac{-1}{15(x^3-3x+1)^5} + c.$$

Example

Evaluate the integral $\int \frac{x^2 - 1}{(x^3 - 3x + 1)^6}\, dx$.

Solution:

Let $u = x^3 - 3x + 1$, then $du = 3(x^2 - 1)\, dx$. By substitution, we have

$$\frac{1}{3} \int u^{-6}\, du = \frac{1}{3}\, \frac{1}{-5u^5} + c = \frac{-1}{15(x^3 - 3x + 1)^5} + c.$$

The upcoming corollary simplifies the process of the substitution method for some functions.

Corollary

If $\int f(x)\, dx = F(x) + c$, *then for any* $a \neq 0$,

$$\int f(ax \pm b)\, dx = \frac{1}{a} F(ax \pm b) + c.$$

Example

Evaluate the integral.

1. $\int \sqrt{2x-5}\; dx$
2. $\int \cos\,(3x+4)\; dx$

Example

Evaluate the integral.

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

From the corollary, we have

1. $\int \sqrt{2x-5}\ dx = \frac{1}{2}\frac{(2x-5)^{3/2}}{3/2} + c = \frac{(2x-5)^{3/2}}{3} + c.$

Example

Evaluate the integral.

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

From the corollary, we have

1. $\int \sqrt{2x-5}\ dx = \frac{1}{2}\frac{(2x-5)^{3/2}}{3/2} + c = \frac{(2x-5)^{3/2}}{3} + c.$
2. $\int \cos\ (3x+4)\ dx = \frac{1}{3}\sin\ (3x+4) + c.$