MATH107 Vectors and Matrices

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function of several variables

Let $z = f(x, y)$ be function of two variable x and y. x, y are independent variables, z is dependent variable. Domain is ordered pair (x, y) and values of $f(x, y)$ is called Range.

Examples

(1) Find the domain for the graph of the function
$f(x, y) = \sqrt{9 - x^2 - y^2}$.

(2) Given that $f(x, y) = 4 + \sqrt{x^2 - y^2}$, find $f(1, 0)$, $f(5, 3)$ and $f(4, -2)$. Sketch the domain of function.

(3) Find the domain for the graph of the function $f(x, y) = \frac{\sqrt{x^2+y^2-16}}{y}$.

Limit Notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{or} \quad f(x,y) = L \quad \text{as } (x,y) \to (a,b)$$

Examples

(1) Find the limit if exist

(a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 - xy + y^2}{x^2 + 2xy - 2y + x}.$

(b) $\displaystyle\lim_{(x,y)\to(2,3)} x^2 + xy + y^2$

(d) $\displaystyle\lim_{(x,y)\to(2,-3)} x^3 - 4xy^2 + 5y - 7$

(c) $\displaystyle\lim_{(x,y)\to(3,4)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}.$

Two path Rule

If two different paths to a point $P(a, b)$ produce two different limiting values for f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Examples

(1) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

(2) Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

Continuity

A function f of two variables is continuous at an interior point (a,b) of its domain if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Note

(1) Polynomial functions are continuous throghout the entire $xy-$plane.
(2) Rational functions are continuous except at points where the denominator is zero.

Definition: Partial Derivatives

Let f be a function of two variables. The first partial derivatives of f with respect to x and y are the function f_x and f_y, such that

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation

if $w = f(x, y)$, then

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}$$

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial w}{\partial x} = w_x$$

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \frac{\partial w}{\partial y} = w_y.$$

Examples

(1) Find the first partial derivative for

(a) $f(x, y) = 3x^4y^3 - 4x^2 + 4y^3 + 5$,

(b) if $f(x, y) = x^2y^3z^4 + 2x - 5yz$.

Second Partial derivatives

$$\frac{\partial}{\partial x} f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$$
$$\frac{\partial}{\partial y} f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$$
$$\frac{\partial}{\partial x} f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial}{\partial y} f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$$

Note

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Examples

(1) Find the second partial derivative of f if
$f(x, y) = xy^4 - 2x^2y^3 + 4x^2 - 3y$.
(2) Verify that $w_{xy} = w_{yx}$, if $w(x, y) = x^3e^{-2y} + y^{-2}\cos x$.
(3) Find f_{xyz} if $f(x, y, z) = \sqrt{x^2 + y^3 + z^4}$.

Increments and differentials

Increments

Let $w = f(x, y)$, and let Δx and Δy be increments of x and y, respectively. The increment Δw of $w = f(x, y)$ is

$$\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Example 1

Let $w = f(x, y) = 3x^2 - xy$. (a) If Δx and Δy are increments of x and y, find Δw.
(b) Use Δw to calculate the change in $f(x, y)$ if (x, y) changes from (1,2) to (1.01,1.98).

Increments and differentials

Differentials

Let $w = f(x, y)$, and let Δx and Δy be increments of x and y, respectively.

(i) The differential dx and dy of the independent variables x and y are

$$dx = \Delta x \text{ and } dy = \Delta y.$$

(ii) The differential dw of the independent variables w is

$$dw = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy.$$

Example 2

If $w = f(x, y) = 3x^2 - xy$, find dw and use it to approximate the change in w if (x,y) changes from (1,2) to (1.01,1.98). How does this compare with the exact change in w?

Increments and differentials

Note

If $w = f(x, y, z, t)$,then total differential

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial t}dt.$$

Ex 1: Find the total differential of function

$$w = x^2z + 4yt^3 - xz^2t$$

Ex 2: Use differential to approximate the change in

$$f(x, y) = x^2 - 2xy + 3y$$

if (1,2) changes to (1.03,1.99).

Chain Rule

Chain Rule

If $w = f(u, v)$, with $u = f(x, y)$, $v = f(x, y)$, and if f, g and h are differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Ex(1) Use Chain rule to find:

(a) $\frac{\partial w}{\partial p}$ and $\frac{\partial w}{\partial q}$ if $w = r^3 + s^2$, with $r = pq^2$, $s = p^2 \sin q$.

(b) $\frac{\partial w}{\partial z}$ if $w = r^2 + sv + t^3$, with $r = x^2 + y^2 + z^2$, $s = xyz$, $v = xe^y$, $t = yz^2$.

(c) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = pq + qw$, with $p = 2x - y$, $q = x - 2y$, $w = -2x + 2y$.

(d) $\frac{\partial p}{\partial r}$ if $p = u^2 + 3v^2 - 4w^2$, with $u = x - 3y + 2r - s$, $v = 2x + y - r + 2s$, $w = -x + 2y + r + s$.

Implicit Partial differentiation

Theorem 1

If an equation $F(x, y) = 0$ determines, implicitly, a differentiable function f of one variable x such as $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

Ex(2) Use partial derivatives to find $\frac{dy}{dx}$
(a) if $2x^3 + x^2y + y^3 = 1$.
(b) if $x^4 + 2x^2y^2 - 3xy^3 + 2x = 0$.

Implicit Partial differentiation

Theorem 2

If an equation $F(x, y, z) = 0$ determines an implicit a differentiable function f of two variables x and y such as $z = f(x, y)$, then

$$\frac{dz}{dx} = -\frac{F_x(x, y)}{F_z(x, y)}$$

$$\frac{dz}{dy} = -\frac{F_y(x, y)}{F_z(x, y)}$$

Ex(2) Use partial derivatives to find $\frac{dz}{dx}$ and $\frac{dz}{dy}$
(a) if $2xz^3 + 3yz^2 + x^2y^2 + 4z = 0$.
(b) if $x^2 + y^2 + z^3 = 9$.