MATH107 Vectors and Matrices

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Properties of Determinantial Function

1. If A is a $n \times n$ matrix $\mathbf{det(kA) = k^n det(A)}$.
2. $\mathbf{det(A + B) \neq det(A) + det(B)}$,
3. $\mathbf{det(AB) = det(A).det(B)}$,
4. $\mathbf{det(A^{-1}) = \frac{1}{det(A)}}$.
5. A sqaure matrix is invertible if and only if $\mathbf{det(A) \neq 0}$, and
6. $\mathbf{det(A^t) = det(A)}$.

Example 1: Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and $det(A) = -7$.

Find (a) $det(3A)$, (b) $det(2A)^{-1}$, (c) $det(2A^{-1})$ and (d) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

Example 2: Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Example 3: Show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

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Minors and co-factors of matrix

Minors: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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Cofactors:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

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Example 1

Find minors and cofactors of matrix A and the its determinant by method of cofactor. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5,\ M_{12} = \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = -6,\ M_{13} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2,$

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$M_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1,\ M_{32} = \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -4,\ M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,$

Cofactors:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C = \begin{bmatrix} -5 & 6 & 2 \\ 7 & -11 & 5 \\ 1 & 4 & -3 \end{bmatrix}$$

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Inverse by method of Cofactors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ det(A) \neq 0$$

Step 1: Find matrix of cofactors $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Step 2: Find adjoint of matrix A, $adj(A) = C^t$

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^t$$

Step 3: If is an invertible matrix, $det(A) \neq 0$ then,

$$A^{-1} = \frac{1}{det(A)}[adj(A)]$$

Cramer's Rule

If A is $n \times n$ matrix with $det(A) \neq 0$, then the linear system $AX = B$ has a unique solution $X = (x_j)$ given by

$$x_j = \frac{det(A_j)}{det(A)}, j = 1, 2 \ldots, n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.
Example: Use Cramer's rule to solve

$$\begin{aligned} 4x + 5y &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$