MATH107 Vectors and Matrices

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Definition 1

The dot product $a.b$ of $a =< a_1, a_2, a_3 >$ and $b =< b_1, b_2, b_3 >$ is $a.b = a_1b_1 + a_2b_2 + a_3b_3$.

Properties of dot product

(1) $a.a = \|a\|^2$
(2) $a.b = b.a$ (commutative)
(3) $a.(b + c) = a.b + a.c$ (distributive)
(4) $(ma).b = m(a.b) = a.(mb)$
(5) $0.a = 0$
(6) $a.b = 0$ if $a = 0$ or $b = 0$ or $\theta = \frac{\pi}{2}$

Definition 2

The dot product of two vectors a and b is scalar $a.b = \|a\|\|b\| \cos\theta$, where θ is angle between the vectors such that $0 \leqslant \theta \leqslant \pi$.
$\cos\theta = \frac{a.b}{\|a\|\|b\|} \Rightarrow \theta = \cos^{-1} \frac{a.b}{\|a\|\|b\|}$.

Notes

(1) If vector a and b are parallel, then $b = ca$.
(2) If $a =< a_1, a_2, a_3 >$ and $b =< b_1, b_2, b_3 >$ are parallel, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.
(3) Two nonzero vectors a and b are orthogonal if and only if $a.b = 0$
(4) $i.j = j.i = 0$, $j.k = k.j = 0$, $k.i = i.k = 0$

Example 1: Find the angle between $a = i - 7j + 4k, b = 5i - k$.
Example 2: Show that $a = 3i - 2j + k$ and $b = 4i + 5j - 2k$ are orthognal.

Component of a along b

Let a and b vectors in V_3 with $b \neq 0$. The component of a along b is $Comp_b^a = \frac{a.b}{\|b\|}$

Projection of a on b

Vector projection of a onto b is $(Comp_b^a).\frac{b}{\|b\|} = (\frac{a.b}{\|b\|})(\frac{b}{\|b\|})$

Example 1: Let $a = 2i + 3j - 4k$ and $b = i + j + 2k$.
Find (1) $Comp_b^a$ (2) $Comp_a^b$ (3) $Proj_b^a$

Work done

The work done by a constant force $\overrightarrow{PQ}$ as its point of application moves along the vector $\overrightarrow{PR}$ is $\overrightarrow{PQ}.\overrightarrow{PR}$

Example 2: Find work done by constant force $F = 2i + 4j + k$ if its point of application moves form $P(1, 1, 3)$ to $Q(4, 6, 2)$.

Direction angles/direction cosines

The direction angles of nonzero vector $a < a_1, a_2, a_3 >$ are the angles α, β and γ with the base vector i, j and k respectively.
The cosine of these angles $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosine of vector a defined as
$\cos\alpha = \frac{a.i}{\|a\|\|i\|} = \frac{a_1}{\|a\|}$,
$\cos\beta = \frac{a.j}{\|a\|\|j\|} = \frac{a_2}{\|a\|}$ and
$\cos\gamma = \frac{a.k}{\|a\|\|k\|} = \frac{a_3}{\|a\|}$.

Note $\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1$.
Example 1: Find the direction cosines and direction angles of the vector $a = 2i + 3j + 4k$ and also show that $\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1$.

Definition 1

The vector product $a \times b$ of $a =< a_1, a_2, a_3 >$ and $b =< b_1, b_2, b_3 >$ is

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} i.$$

Definition 2

The vector product of a and b is the vector

$$a \times b = (\|a\| \|b\| \sin\theta)\mathbf{n}$$

where θ is the angle between the vectors such that $0 \leqslant \theta \leqslant \pi$ and **n** is a unit vector perpendicular to the plane of a and b with direction given by the right-hand rule.

Properties of vector product

(1) $a \times b = -(b \times a)$
(2) $a \times (b + c) = (a \times b) + (a \times c)$
(3) $(a + b) \times c = (a \times c) + (b \times c)$ (distributive)
(4) $(ma) \times b = m(a \times b) = a \times (mb)$
(5) $(a \times b).c = a.(b \times c)$ (Triple scalar product)
(6) $a \times (b \times c) = (a.c)b - (a.b)c$(Triple vector product)

Notes:

1- $a \times b$ is orthogonal to both a and b, i.e $(a \times b).a = 0$ or $(a \times b).b = 0$

2- a and b are parallel if $a \times b = 0$.

3- a and b are orthogonal if $a.b = 0$.

4- The magnitude of $a \times b$ equals the area of the parallelogram. It means that $\|a \times b\| = \|a\|\|b\| \sin\theta$.

5- Area of triangle $= \frac{\|a \times b\|}{2}$.

Properties of i, j, k

(1) $i \times j = k$, $j \times k = i$ and $k \times i = j$
(2) $j \times i = -k$, $k \times j = -i$ and $i \times k = -j$
(3) $i \times i = j \times j = k \times k = 0$

Distance of a point R to line l

$d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|}$.

Volume of a box

$V = |(a \times b).c|$.