A Useful Theorem about Limits

Theorem 2.1.4 *If*

$$\lim_{x \to x_0} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to x_0} g(x) = L_2, \tag{2.1.9}$$

then

$$\lim_{x \to x_0} (f + g)(x) = L_1 + L_2, \tag{2.1.10}$$

$$\lim_{x \to x_0} (f - g)(x) = L_1 - L_2, \tag{2.1.11}$$

$$\lim_{x \to x_0} (fg)(x) = L_1 L_2, \tag{2.1.12}$$

and, if $L_2 \neq 0$, (2.1.13)

$$\lim_{x \to x_0} \left(\frac{f}{g} \right)(x) = \frac{L_1}{L_2}. \tag{2.1.14}$$

Proof From (2.1.9) and Definition 2.1.2, if $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$|f(x) - L_1| < \epsilon \tag{2.1.15}$$

if $0 < |x - x_0| < \delta_1$, and a $\delta_2 > 0$ such that

$$|g(x) - L_2| < \epsilon \tag{2.1.16}$$

if $0 < |x - x_0| < \delta_2$. Suppose that

$$0 < |x - x_0| < \delta = \min(\delta_1, \delta_2), \tag{2.1.17}$$

so that (2.1.15) and (2.1.16) both hold. Then

$$\begin{aligned} |(f \pm g)(x) - (L_1 \pm L_2)| &= |(f(x) - L_1) \pm (g(x) - L_2)| \\ &\le |f(x) - L_1| + |g(x) - L_2| < 2\epsilon, \end{aligned}$$

which proves (2.1.10) and (2.1.11).

To prove (2.1.12), we assume (2.1.17) and write

$$\begin{aligned} |(fg)(x) - L_1L_2| &= |f(x)g(x) - L_1L_2| \\ &= |f(x)(g(x) - L_2) + L_2(f(x) - L_1)| \\ &\le |f(x)||g(x) - L_2| + |L_2||f(x) - L_1| \\ &\le (|f(x)| + |L_2|)\epsilon \quad \text{(from (2.1.15) and (2.1.16))} \\ &\le (|f(x) - L_1| + |L_1| + |L_2|)\epsilon \\ &\le (\epsilon + |L_1| + |L_2|)\epsilon \quad \text{from (2.1.15)} \\ &\le (1 + |L_1| + |L_2|)\epsilon \end{aligned}$$

if $\epsilon < 1$ and x satisfies (2.1.17). This proves (2.1.12).

To prove (2.1.14), we first observe that if $L_2 \ne 0$, there is a $\delta_3 > 0$ such that

$$|g(x) - L_2| < \frac{|L_2|}{2},$$

so

$$|g(x)| > \frac{|L_2|}{2} \tag{2.1.18}$$

if

$$0 < |x - x_0| < \delta_3.$$

To see this, let $L = L_2$ and $\epsilon = |L_2|/2$ in (2.1.4). Now suppose that

$$0 < |x - x_0| < \min(\delta_1, \delta_2, \delta_3),$$

so that (2.1.15), (2.1.16), and (2.1.18) all hold. Then

$$
\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \frac{L_1}{L_2}\right| &= \left|\frac{f(x)}{g(x)} - \frac{L_1}{L_2}\right| \\
&= \frac{|L_2 f(x) - L_1 g(x)|}{|g(x)L_2|} \\
&\le \frac{2}{|L_2|^2}|L_2 f(x) - L_1 g(x)| \\
&= \frac{2}{|L_2|^2}\,|L_2[f(x) - L_1] + L_1[L_2 - g(x)]| \quad \text{(from (2.1.18))} \\
&\le \frac{2}{|L_2|^2}\,[|L_2||f(x) - L_1| + |L_1||L_2 - g(x)|] \\
&\le \frac{2}{|L_2|^2}(|L_2| + |L_1|)\epsilon \quad \text{(from (2.1.15) and (2.1.16))}.
\end{aligned}
$$

This proves (2.1.14). ◻

Successive applications of the various parts of Theorem 2.1.4 permit us to find limits without the ϵ–δ arguments required by Definition 2.1.2.