CHAPTER 1

SYSTEMS OF LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

1 Introduction

In this chapter we give an introduction to linear systems of differential equations. The general form of such system is

$$\frac{dX}{dt}(t) = A(t)X(t) + F(t),$$

where $X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$, $F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$.

If $F = 0$, the system is called homogeneous, otherwise it is non-homogeneous.

Example 1 :

The motion of a spring-mass system from is described by the equation

$$u''(t) + au'(t) + bu(t) = 0.$$

This second order equation can be converted into a system of first order equations by letting $x = u$ and $y = u'$. Thus $\begin{cases} x' &= & y \\ y' &= & -bx - ay \end{cases}$.

Example 2 :

The equation $u''(t) - 3u'(t) + 2u(t) = \sin t$ can be written as system of first order equations by letting $x = u$ and $y = u'$. Thus $\begin{cases} x' &= & y \\ y' &= & -2x + 3y + \sin t \end{cases}$.

Theorem 1.1: Existence and Uniqueness Solution

If the matrices $A(t)$ and $F(t)$ are continuous on an open interval I. For all $t_0 \in I$, there exists a unique solution of the initial value problem $X'(t) = A(t)X(t) + F(t)$ and $X(t_0) = X_0$ on the interval.

2 Homogeneous Systems

We consider the homogeneous linear system $X'(t) = A(t)X(t)$.

Theorem 2.1: Superposition Principle

If $X_1, \ldots, X_m$ are solutions of the linear system $X'(t) = A(t)X(t)$ on an interval I, then $a_1X_1 + \ldots + a_mX_m$ is also solution of the linear system on the interval I.

2.1 Linear Dependence and Linear Independence

Definition 2.1:

Let $X_1, \ldots, X_m$ be a set of vectors solutions of the linear system $X'(t) = A(t)X(t)$ on an interval I. We say that the set is linearly dependent, if there exist constants $c_1, \ldots, c_m$, not all zero, such that

$$c_1X_1(t) + \ldots + c_mX_m(t) = 0$$

for all $t \in I$. Otherwise, the set is called linearly independent.

Theorem 2.2: Criterion for Linearly Independent Solutions

Let $X_1(t) = \begin{pmatrix} x_{1,1}(t) \\ \vdots \\ x_{1,n}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{n,1}(t) \\ \vdots \\ x_{n,n}(t) \end{pmatrix}$ be n solutions of the linear system $X'(t) = A(t)X(t)$ on an interval I. Then the set of solution vectors is linearly independent on I if and only if the Wronskian

$$W(X_1, \ldots, X_n) = \begin{vmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{vmatrix} \neq 0,$$

for all $t \in I$.

Proof .

□

Remark 1 :

If $X_1(t) = \begin{pmatrix} x_{1,1}(t) \\ \vdots \\ x_{1,n}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{n,1}(t) \\ \vdots \\ x_{n,n}(t) \end{pmatrix}$ be n vectors functions not necessary solutions of a linear system $X'(t) = A(t)X(t)$ on an interval I. Then the set $X_1, \ldots, X_n$ is linearly independent on I if and only if there exists $t_0 \in I$ such that the Wronskian $W(X_1(t_0), \ldots, X_n(t_0)) \neq 0$.

Definition 2.2: Fundamental Set of Solutions

Any set of n linearly independent solution vectors of the homogeneous system $X'(t) = A(t)X(t)$ on an interval I is said to be a fundamental set of solutions on the interval.

Theorem 2.3: Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous system $X'(t) = A(t)X(t)$ on an interval I.

Theorem 2.4: General Solution for Homogeneous Systems

Let $X_1, \ldots, X_n$ be a fundamental set of solutions of the homogeneous system $X'(t) = A(t)X(t)$ on an interval I. Then the general solution of the system on the interval is $X(t) = c_1X_1(t) + \ldots + c_nX_n(t)$, where the $c_1, \ldots, c_n$ are arbitrary constants.

2.2 Non-Homogeneous Systems

Theorem 2.5: General Solution for Non-Homogeneous Systems

Let X_p be a given solution of the non-homogeneous system on an interval I and let $X_c(t) = c_1X_1(t) + \ldots + c_nX_n(t)$, denote the general solution on the same interval of the associated homogeneous system. Then the general solution of the non-homogeneous system on the interval I is

$$X(t) = X_c(t) + X_p.$$

3 Homogeneous Linear Systems with Constant Coefficients

3.1 Eigenvalues and Eigenvectors

Definition 3.1

A nonzero vector v is an eigenvector of a matrix A if $Av = \lambda v$ for some $\lambda \in \mathbb{R}$. The constant λ is called an eigenvalue of A.

If v is a eigenvector of a matrix A, $Av = \lambda v \iff (A - \lambda I)v = 0$. This will occur exactly when the determinant of $(A - \lambda I)$ is zero. In this case $\det(A - \lambda I)$ is called the characteristic polynomial of A.

Theorem 3.1:

The roots of the characteristic polynomial of A are the eigenvalues of A.

In what follows, we consider only the case where the Matrix A is a constant $(2,2)-$matrix.
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $v = \begin{pmatrix} x \\ y \end{pmatrix}$.
$Av = \lambda v \iff (A - \lambda I)v = 0$. This will occur exactly when the determinant of $(A - \lambda I)$ is zero. In this case $\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc)$ is called the characteristic polynomial of A.

3.2 Changing Coordinates

We consider a linear system $X'(t) = AX(t)$ and the change of coordinates $X = TY$, where T is an invertible matrix. Then X is a solution of the system $X'(t) = AX(t)$ is equivalent that Y is a solution of the system $Y'(t) = T^{-1}ATY(t)$.

3.3 Distinct Real Eigenvalues

Now we assume that the matrix A has two distinct real eigenvalues $\lambda_1 \neq \lambda_2$. Let X_1 and X_2 such that $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_1 X_2$, with $X_1 \neq 0$ and $X_2 \neq 0$. The general solution of the homogenous system is

$$X = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2.$$

3.4 Repeated eigenvalues

Now suppose that A has a single real eigenvalue α. Then the characteristic polynomial of A is $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$, then A has an eigenvalue $\lambda = \frac{1}{2}(a+d)$.
We choose $X_2 \in \ker(A - \lambda I)^2$ and not in $\ker(A - \lambda I)$. We denote $X_1 = (A - \lambda I)X_2$. X_1 is an eigenvector of the matrix A for the eigenvalue λ. The general solution of the linear system of differential equations $X' = AX$ is

$$X = ae^{\lambda t} X_1 + b(te^{\lambda t} X_1 + e^{\lambda t} X_2).$$

3.5 Complex eigenvalues

Now we assume that the matrix A has non real eigenvalues $\lambda = \pm\alpha + i\beta$ with $\beta \neq 0$. Let $X = X_1 + iX_2$ be an eigenvector of the matrix A for the eigenvalue λ. The general solution of the homogenous system is

$$X = c_1 e^{\alpha t} \left(X_1 \cos(\beta t) - X_2 \sin(\beta t) \right) + c_2 e^{\alpha t} \left(X_1 \sin(\beta t) - X_2 \cos(\beta t) \right).$$

Example 3 :
Let $X' = AX$ the system with $A = \begin{pmatrix} -3 & 2 \\ -1 & -5 \end{pmatrix}$.
$\begin{vmatrix} -3 - \lambda & 2 \\ -1 & -5 - \lambda \end{vmatrix} = (\lambda + 4)^2 + 1$. Then $\lambda = -4 \pm i$ are the eigenvalues of A. The

vector $X = \begin{pmatrix} -2 \\ 1-\mathrm{i} \end{pmatrix}$ is an eigenvector of the matrix.
The general solution of the system is $X = c_1 e^{-4t}\left(X_1\cos(t) - X_2\sin(t)\right) + c_2 e^{-4t}\left(X_1\sin(t) - X_2\cos(t)\right)$, with $X_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

4 The Laplace Transform Method of Solving Systems of Linear Equations

The method of Laplace transforms, in addition to solving individual linear differential equations, can also be used to solve systems of linear differential equations.

Example 4 :

Consider the system of linear differential equations: $X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X$ with $X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This system is rewritten into the explicit form as follows:

$$\begin{cases} x'(t) &= x(t) - y(t) \\ y'(t) &= x(t) + y(t) \end{cases}$$

Using Laplace transform, we get

$$\begin{cases} s\mathcal{L}(x)(s) - 2 &= \mathcal{L}(x)(s) - \mathcal{L}(y)(s) \\ s\mathcal{L}(y)(s) - 1 &= \mathcal{L}(x)(s) + \mathcal{L}(y)(s) \end{cases} \Longleftrightarrow \begin{cases} (s-1)\mathcal{L}(x)(s) + \mathcal{L}(y)(s) &= 2 \\ -\mathcal{L}(x)(s) + (s-1)\mathcal{L}(y)(s) &= 1 \end{cases}$$

Then $\mathcal{L}(x)(s) = \dfrac{2s-3}{(s-1)^1+1} = \dfrac{2(s-1)-1}{(s-1)^1+1}$ and $\mathcal{L}(y)(s) = \dfrac{(s-1)+2}{(s-1)^1+1}$.
Taking the Laplace inverse operator, we get

$$x(t) = e^t(2\cos(t) - \sin(t)), \quad y(t) = e^t(\cos(t) + 2\sin(t)).$$

Example 5 :

Consider the system of linear differential equations: $X' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X$ with $X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. This system is rewritten into the explicit form as follows:

$$\begin{cases} x'(t) &= -4x(t) - 2y(t) \\ y'(t) &= 3x(t) + y(t) \end{cases}$$

Using Laplace transform, we get

$$\begin{cases} s\mathcal{L}(x)(s) - 2 &= -4\mathcal{L}(x)(s) - 2\mathcal{L}(y)(s) \\ s\mathcal{L}(y)(s) - 3 &= 3\mathcal{L}(x)(s) + \mathcal{L}(y)(s) \end{cases} \Longleftrightarrow \begin{cases} (s+4)\mathcal{L}(x)(s) + 2\mathcal{L}(y)(s) &= 2 \\ -3\mathcal{L}(x)(s) + (s-1)\mathcal{L}(y)(s) &= 3 \end{cases}$$

Then $\mathcal{L}(x)(s) = \dfrac{2s-8}{(s+1))(s+2)} = -\dfrac{10}{s+1} + \dfrac{12}{s+2}$ and
$\mathcal{L}(y)(s) = \dfrac{3s+18}{(s+1)(s+2)} = \dfrac{15}{s+1} - \dfrac{12}{s+2}$.

Taking the Laplace inverse operator, we get

$$x(t) = -10e^{-t} + 12e^{-2t}, \quad y(t) = 15e^{-t} - 12e^{-2t}.$$

Example 6 :

Consider the system of linear differential equations: $X' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ 3e^{-2t} \end{pmatrix}$ with $X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. This system is rewritten into the explicit form as follows:

$$\begin{cases} x'(t) & = & -4x(t) - 2y(t) + e^{-t} \\ y'(t) & = & 3x(t) + y(t) + 3e^{-2t} \end{cases}$$

Using Laplace transform, we get

$$\begin{cases} s\mathcal{L}(x)(s) - 2 & = & -4\mathcal{L}(x)(s) - 2\mathcal{L}(y)(s) + \frac{1}{s+1} \\ s\mathcal{L}(y)(s) - 3 & = & 3\mathcal{L}(x)(s) + \mathcal{L}(y)(s) + \frac{3}{s+2} \end{cases} \Longleftrightarrow \begin{cases} (s+4)\mathcal{L}(x)(s) + 2\mathcal{L}(y)(s) & = & 2 + \frac{1}{s+1} \\ -3\mathcal{L}(x)(s) + (s-1)\mathcal{L}(y)(s) & = & 3 + \frac{3}{s+2} \end{cases}$$

Then $\mathcal{L}(x)(s) = -\dfrac{13}{s+1} + \dfrac{15}{s+2} - \dfrac{2}{(s+1)^2} + \dfrac{6}{(s+2)^2}$ and
$\mathcal{L}(y)(s) = \dfrac{21}{s+1} - \dfrac{18}{s+2} + \dfrac{3}{(s+1)^2} - \dfrac{6}{(s+2)^2}$.
Taking the Laplace inverse operator, we get
$x(t) = -13e^{-t} + 15e^{-2t} - 2te^{-t} + 6te^{-2t}$,
$y(t) = 21e^{-t} - 18e^{-2t} + 3te^{-t} - 6te^{-2t}$.