King Saud University Mathematics Department M-254
Summer Semester (1st Midterm Exam) 1437-1438 H
Max Marks=25 Time Allowed: 90 Mins.

Question 1: (5)

Which of the following iterations

$(i)\quad x_{n+1} = e^{x_n} - x_n - 1, \quad n \geq 0, \qquad (ii)\quad x_{n+1} = \ln(2x_n + 1), \quad n \geq 0,$

is most suitable to approximate the root of the equation $e^x - 2x = 1$ in the interval $[1, 2]$? Starting with $x_0 = 1.5$, find the second approximation x_2 of the root. Also, compute the error bound for the approximation.

Question 2: (5)

Successive approximations x_n to the desired root are generated by the scheme

$$x_{n+1} = e^{x_n} - 2, \qquad n \geq 0.$$

Find $f(x_n)$ and its derivative $f'(x_n)$ and then use Newton's method to find the second approximation x_2 of the root, starting with $x_0 = 10$.

Question 3: (5)

Show that $\alpha = 1$ is the root for the equation $x^4 - x^3 - 3x^2 = 2 - 5x$. Use quadratic convergent iterative method to find the first approximation of α starting with $x_0 = 0.5$. Compute absolute error.

Question 4: (5)

If $x = \alpha$ is a root of multiplicity 5 of $f(x) = 0$, then show that the rate of convergence of modified Newton's method is at least quadratic.

Question 5: (5)

Find the first approximation for the nonlinear using $(x_0, y_0)^T = (0.5, -0.5)^T$ system

$$\begin{aligned} y &= -\sqrt{x} \\ (x-3)^2 + y^2 &= 5 \end{aligned}$$

Solution of the Midterm I Examination

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Question 1: (5)
Which of the following iterations

$$(i) \quad x_{n+1} = e^{x_n} - x_n - 1, \quad n \geq 0 \quad (ii) \quad x_{n+1} = \ln(2x_n + 1), \quad n \geq 0$$

is most suitable to approximate the root of the equation $e^x - 2x = 1$ in the interval $[1, 2]$? Starting with $x_0 = 1.5$, find the second approximation x_2 of the root. Also, compute the error bound for the approximation.

Solution. Since $f(x) = e^x - 2x - 1$, and $f(1).f(2) = (-0.2817)(2.3891) < 0$, then the solution we seek is in the interval $[1, 2]$.

(i) For the first scheme, $g(x) = e^x - x - 1$ is continuous in $[1, 2]$ but both $g(1) = 0.718$ and $g(2) = 4.39$ are not in $[1, 2]$. Also, $g'(x) = e^x - 1$, which is greater than unity throughout the interval $[1, 2]$. So by the Fixed-Point Theorem, this iteration will fail to converge.

(ii) For the second scheme, we are given $g(x) = \ln(2x+1)$, which is continuous in $[1, 2]$ and $g(1) = \ln(3) = 1.0986123$ and $g(2) = \ln(5) = 1.6094379$ both lie in the interval $[1, 2]$. Since g is increasing function of x, and $g(x) \in [1, 2]$, for all $x \in [1, 2]$. Also, we have $g'(x) = 2/(2x + 1) < 1$, for all x in the given interval $[1, 2]$. So from Fixed-Point Theorem, this g(x) has a unique fixed-point. For finding the second approximation of the root lying in the interval $[1, 2]$, using the given initial approximation $x_0 = 1.5$,

$$x_1 = \ln(2(1.5) + 1) = 1.386294, \quad \text{and} \quad x_2 = \ln(2x_1 + 1) = 1.327761.$$

To compute the error bound,

$$|\alpha - x_2| \leq \frac{k^2}{1 - k}|x_1 - x_0|,$$

we need $k_1 = |g'(1)| = 0.66667$, $k_2 = |g'(2)| = 0.40$ and $k = \max\{k_1, k_2\} = 0.667$, therefore, the error bound for our approximation will be as follows:

$$|\alpha - x_2| \leq \frac{(0.667)^2}{1 - 0.667}|1.386294 - 1.5| == 0.1516.$$

Question 2: (5)

Successive approximations x_n to the desired root are generated by the scheme

$$x_{n+1} = e^{x_n} - 2, \qquad n \geq 0$$

Find $f(x_n)$ and its derivative $f'(x_n)$ and then use Newton's method to find the first approximation of the root, starting with $x_0 = 10$.

Solution: Given $x = e^x - 2 = g(x)$, and it can be written as

$$g(x) - x = e^x - 2 - x = 0,$$

so

$$f(x) = e^x - 2 - x, \quad f'(x) = e^x - 1.$$

Thus

$$f(x_n) = e^{x_n} - x_n - 2, \; f'(x_n) = e^{x_n} - 1.$$

and the Newton's method gets the form

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n - 2}{e^{x_n} - 1}, \quad n \geq 0.$$

Using it to find first approximation x_1, with $x_0 = 10$

$$x_1 = x_0 - \frac{e^{x_0} - x_0 - 2}{e^{x_0} - 1} = 10 - \frac{e^{10} - 10 - 2}{e^{10} - 1} = 9.0005.$$

Question 3: (5)

Show that $\alpha = 1$ is the root for the equation

$$x^4 - x^3 - 3x^2 = 2 - 5x.$$

Use quadratic convergent iterative method to find the first approximation of α starting with $x_0 = 0.5$. Compute absolute error.

Solution: Since $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$. First we show that $\alpha = 1$ is the zero of the given function as

$$f(\alpha) = f(1) = (1)^4 - (1)^3 - 3(1)^2 + 5(1) - 2 = 0.$$

To check whether it is *simple* or *multiple* zero of $f(x)$, we do the following

$$f'(x) = 4x^3 - 3x^2 - 6x + 5 \qquad \text{and} \qquad f'(\alpha) = f'(1) = 4 - 3 - 6 + 5 = 0,$$

which means that $\alpha = 1$ is the multiple zero of the given function. To find its order of multiplicity, we do

$$f''(x) = 12x^2 - 6x - 6 \qquad \text{and} \qquad f''(\alpha) = f''(1) = 12x^2 - 6x - 6 = 0,$$

$$f'''(x) = 24x - 6 \qquad \text{and} \qquad f'''(\alpha) = f'''(1) = 24 - 6 = 18 \neq 0,$$

hence $\alpha = 1$ is a zero of multiplicity 3 of the given function. Then

$$x_{n+1} = x_n - m\frac{(x_n^4 - x_n^3 - 3x_n^2 + 5x_n - 2)}{(4x_n^3 - 3x_n^2 - 6x_n + 5)} = 0.5 - 3\frac{(-0.3125)}{(1.75)} = 0.6786,$$

and the possible absolute error is

$$\text{Absolute Error} = |1 - 0.6786| = 0.3214.$$

Question 4: (5)

If $x = \alpha$ is a root of multiplicity 5 of $f(x) = 0$, then show that the rate of convergence of modified Newton's method is at least quadratic.

Solution: The modified Newton's iterative formula is:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = g(x_n), \quad n \geq 0.$$

and in the fixed-point iteration can be obtained as follows: Since the equation $f(x) = 0$ has multiple root of multiplicity 5, so $f(x)$ can be written as

$$f(x) = (x-\alpha)^5 h(x), \quad f'(x) = 5(x-\alpha)^4 h(x) + (x-\alpha)^5 h'(x).$$

Substituting the values of the $f(x)$ and $f'(x)$ in the above function iteration form, we get

$$g(x) = x - \frac{5(x-\alpha)^5 h(x)}{(5(x-\alpha)^4 h(x) + (x-\alpha)^5 h'(x))} = x - \frac{5(x-\alpha)h(x)}{(5h(x) + (x-\alpha)h'(x))}.$$

Then

$$\begin{aligned} g'(x) = 1 \quad - \quad & 5\{([5h(x) + (x-\alpha)][h(x) + (x-\alpha)h'(x)] - [(x-\alpha)h(x)] \\ & [5h'(x) + h'(x) + (x-\alpha)h''(x)])\}/([5h(x) + (x-\alpha)h'(x)]^2). \end{aligned}$$

At $x = \alpha$, we have

$$g'(\alpha) = 1 - \frac{[5^2 h^2(\alpha)]}{[5h(\alpha)]^2} = 0.$$

Question 5: (5)

Find the first approximation for the nonlinear system

$$\begin{aligned} y &= -\sqrt{x} \\ (x-3)^2 + y^2 &= 5 \end{aligned}$$

using Newton's method, starting with initial approximation $(x_0, y_0)^T = (0.5, -0.5)^T$.

Solution: Solving the given nonlinear system using the Newton's method, we do the following:

$$\begin{aligned} f_1(x,y) &= \sqrt{x} + y, & f_{1x} &= \frac{1}{2\sqrt{x}}, & f_{1y} &= 1 \\ f_2(x,y) &= (x-3)^2 + y^2 - 5, & f_{2x} &= 2(x-3), & f_{2y} &= 2y. \end{aligned}$$

At the given initial approximation $x_0 = 0.5$ and $y_0 = -0.5$, we get

$$f_1(0.5, -0.5) = 0.205, \quad \frac{\partial f_1}{\partial x} = f_{1x} = 0.707, \quad \frac{\partial f_1}{\partial y} = f_{1y} = 1$$

$$f_2(0.5, -0.5) = 1.5, \quad \frac{\partial f_1}{\partial x} = f_{2x} = -5.0, \quad \frac{\partial f_2}{\partial y} = f_{2y} = -1$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0.707 & 1 \\ -5 & -1 \end{pmatrix}$$

and one can find its inverse as

$$J^{-1} = \frac{1}{4.293} \begin{pmatrix} -1 & -1 \\ 5 & 0.707 \end{pmatrix}$$

Substituting all these values in the Newton's formula to get the first approximation as follows

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} - \begin{pmatrix} -0.233 & -0.233 \\ 1.165 & 0.165 \end{pmatrix} \begin{pmatrix} 0.205 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 0.898 \\ -0.987 \end{pmatrix}$$