Solution of the Final Examination

King Saud University	Mathematics Department	Math-254
Summer Semester	Final Examination	1437-1438 H
Max Marks=40	Time Allowed:	180 Mins.

Question 1: (5)

Show that the Secant method for finding approximation of the cubic root of a positive number N is

$$x_{n+1} = \frac{x_n x_{n-1}(x_n + x_{n-1}) + N}{x_n^2 + x_n x_{n-1} + x_{n-1}^2}, \quad n \geq 1.$$

Then use it to find the second approximation of the cubic root of 27, using $x_0 = 2.5$ and $x_1 = 3.5$. Compute absolute error.

Solution. We shall compute $x = N^{1/3}$ by finding a positive root for the nonlinear equation

$$x^3 - N = 0,$$

where $N > 0$ is the number whose root is to be found. If $f(x) = 0$, then $x = \alpha = N^{1/3}$ is the exact zero of the function

$$f(x) = x^3 - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})} = x_n - \frac{(x_n - x_{n-1})(x_n^3 - N)}{(x_n^2 - N) - (x_{n-1}^3 - N)}.$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^3 - N)}{(x_n^3 - x_{n-1}^3)} = x_n - \frac{(x_n - x_{n-1})(x_n^3 - N)}{(x_n - x_{n-1})(x_n^2 + x_n x_{n-1} + x_{n-1}^2)}.$$

$$x_{n+1} = \frac{(x_n^3 + x_n^2 x_{n-1} + x_n x_{n-1}^2 - x_n^3 + N)}{(x_n^2 + x_n x_{n-1} + x_{n-1}^2)} = \frac{x_n x_{n-1}(x_n + x_{n-1}) + N}{x_n^2 + x_n x_{n-1} + x_{n-1}^2}.$$

Taking $n = 1$, using $N = 27$ and $x_0 = 2.5,\ x_1 = 3.5$, gives

$$x_2 = \frac{x_1 x_0(x_1 + x_0) + N}{x_1^2 + x_1 x_0 + x_0^2} = 2.917,$$

and taking $n = 21$, using $N = 27$ and $x_1 = 3.5,\ x_2 = 2.917$, gives

$$x_3 = \frac{x_2 x_1(x_2 + x_1) + N}{x_2^2 + x_2 x_1 + x_1^2} = 2.987.$$

$$\text{Absolute Error} = |3 - 2.987| = 0.013.$$

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Question 2: (5)

Develop the iterative formula

$$x_{n+1} = \frac{2x_n^3 - ax_n^2 - c}{3x_n^2 - 2ax_n + b}, \qquad n \geq 0,$$

for the approximate roots of the cubic equation $x^3 - ax^2 + bx + c = 0$ using the Newton's method. Then use the formula to find the second approximation of the positive root $\alpha = 1$ of the equation $x^3 + 4x^2 = 6 - x$, starting with $x_0 = 0.8$. Show that rate of convergence of the iterative formula is at least quadratic.

Solution: Given

$$f(x) = x^3 - ax^2 + bx + c,$$

therefore, we have

$$f(x_n) = x_n^3 - ax_n^2 + bx_n + c, \quad \text{and} \quad f'(x_n) = 3x_n^2 - 2ax_n + b.$$

Using these functions values in the Newton's iterative formula [illegible], we have

$$x_{n+1} = x_n - \frac{x_n^3 - ax_n^2 + bx_n + c}{3x_n^2 - 2ax_n + b} = \frac{2x_n^3 - ax_n^2 - c}{3x_n^2 - 2ax_n + b}, \quad n \geq 0.$$

Finding the first two approximations of the positive root of $x^3 + 4x^2 = 6 - x$ using the initial approximation $x_0 = 0.8$ and $a = -4, b = 1, c = -6$, we use the above formula by taking $n = 0, 1, 2$ as follows

$$x_1 = \frac{2x_0^3 + 4x_0^2 + 6}{3x_0^2 + 8x_0 + 1} = 1.0283$$

$$x_2 = \frac{2x_1^3 + 4x_1^2 + 6}{3x_1^2 + 8x_1 + 1} = 1.0005,$$

are the possible two approximations. Since the given iteration is

$$x_{n+1} = \frac{2x_n^3 + 4x_n^2 + 6}{3x_n^2 + 8x_n + 1} = g(x_n), \quad \text{which gives ,} \quad g(x) = \frac{2x^3 + 4x^2 + 6}{3x^2 + 8x + 1}.$$

The first derivative of $g(x)$ can be found as

$$g'(x) = \frac{6x^4 + 32x^3 + 38x^2 - 28x - 48}{(3x^2 + 8x + 1)^2}.$$

To find the order of convergence of the iteration, we have to check the derivative $g'(x)$ at fixed-point $x = \alpha = 1$, if it is equal to zero, then order is at least quadratic,

$$g'(1) = \frac{0}{144} = 0.$$

Question 3: (5)

Consider the following system

$$\begin{array}{rcrcrcl} 4x_1 & + & 3x_2 & & & = & 1 \\ 2x_1 & + & 4x_2 & - & x_3 & = & 2 \\ & - & x_2 & + & 4x_3 & = & 3 \end{array}$$

If $x^{(0)} = [0, 0.5, 0.5]^T$, then compute an error bound $\|x - x^{(10)}\|$ for the approximation using Jacobi method. Find the number of iterations k if $\|x - x^{(k)}\| \leq 10^{-6}$.

Solution: Since $k = 10$, we know that error bound formula for the Jacobi method is

$$\|\mathbf{x} - \mathbf{x^{(10)}}\| \leq \frac{\|T_J\|^{10}}{1 - \|T_J\|} \|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\|.$$

So we have to find $\|T_J\|$ and the first approximation $\mathbf{x}^{(1)}$. Since the Jacobi iteration matrix is defined as

$$T_J = -D^{-1}(L + U),$$

$$T_J = -\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & -3 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{4} & 0 \\ -\frac{2}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

Then the l_∞ norm of the matrix T_J is $\|T_J\|_\infty = \max\left\{\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right\} = \frac{3}{4}$. Now to find the first approximation using Jacobi method, we will the following formula

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}\Big[1 - 3x_2^{(k)} \Big] \\ x_2^{(k+1)} &= \frac{1}{4}\Big[2 - 2x_1^{(k)} + x_3^{(k)}\Big] \\ x_3^{(k+1)} &= \frac{1}{4}\Big[3 + x_2^{(k+1)}\Big] \end{aligned}$$

Starting with $x^{(0)} = [0, 0.5, 0.5]^T$ and for $k = 0$, we obtain $\mathbf{x^{(1)}} = [-0.1250, 0.6250, 0.8750]^T$.

$$\|\mathbf{x} - \mathbf{x^{(10)}}\| \leq \frac{(0.75)^{10}}{0.25}(0.375) = 0.0845,$$

the required an error bound. To find the number of iterations k, we use the formula

$$\|\mathbf{x} - \mathbf{x^{(k)}}\| \leq \frac{\|T_J\|^k}{1 - \|T_J\|} \|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\| \leq 10^{-6}.$$

$$\frac{(3/4)^k}{1/4}(0.375) \leq 10^{-6}, \quad \text{or} \quad (3/4)^k \leq \frac{(0.25 \times 10^{-6})}{0.375}.$$

Taking ln on both sides, we obtain, $k \geq 49.4300$, or $k = 50$. •

Question 4: (5)

Construct the divided difference table for $f(x) = \ln(x+2)$, using $x = 0, 1, 2, 3$. Find the approximation of $\ln(3.5)$ using cubic Newton divided difference interpolation formula $p_3(x)$ when $p_2(x) = 1.2620$. Compute error bound for the approximation.

Solution. Constructed divided difference table is Since cubic Newton divided difference inter-

Table 1: Divide differences table for the Example ??

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	0	**0.6932**			
1	1	1.0986	**0.4055**		
2	2	1.3863	0.2877	**- 0.0589**	
3	3	1.6094	0.2232	- 0.0323	**0.0089**

polation formula is

$$p_3(x) = p_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2),$$

or

$$p_3(x) = p_2(x) + 0.0089(x - 0)(x - 1)(x - 2),$$

then at $x = 1.5$, we get

$$p_3(1.5) = p_2(1.5) + 0.0089[(1.5 - 0)(1.5 - 1)(1.5 - 2)) = 1.2620 - 0.0033 = 1.2587.$$

Since the error bound for the cubic polynomial $p_3(x)$ is

$$|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)|$$

Taking the fourth derivative of the given function, we have

$$f^{(4)}(x) = \frac{-6}{(x+2)^4},$$

and

$$|f^{(4)}(\eta(x))| = \left|\frac{-6}{(\eta(x)+2)^4}\right|, \qquad \text{for} \quad \eta(x) \in (0, 3).$$

Since

$$\begin{array}{rcl} |f^{(4)}(0)| & = & 0.375 \\ |f^{(4)}(3)| & = & 0.0096 \end{array}$$

so $|f^{(4)}(\eta(x))| \leq \max\limits_{0 \leq x \leq 3} \left|\frac{-6}{(x+2)^4}\right| = 0.375$, and

$$|f(1.5) - p_3(1.5)| \leq (0.5625)(0.375)/24 = 0.0088,$$

which is the required error bound for the approximation $p_3(1.5)$. •

Question 5: (5)

Let $f(x) = \dfrac{2}{x}$ and the points $x_0 = 1, x_1 = 1, x_2 = 1, x_3 = 2$.
Compute the approximation of $f(1.5)$ by using the cubic Newton's interpolating polynomial $p_3(x)$ and find the absolute error.

Solution. Since the cubic Newton's interpolating polynomial has the following form

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_0]+(x-x_0)(x-x_0)f[x_0,x_0,x_1]+(x-x_0)(x-x_0)(x-x_1)f[x_0,x_0,x_1,x_1],$$

$$p_3(x) = f[x_0]+(x-x_0)f'(x_0)+(x-x_0)(x-x_0)\frac{f''(x_0)}{2}+(x-x_0)(x-x_0)(x-x_1)f[x_0,x_0,x_1,x_1],$$

and using the points and $x = 1.5$, we get

$$p_3(1.5) = f(1) + (1.5-1)f'(1) + (1.5-1)(1.5-1)\frac{f''(1)}{2} + (1.5-1)(1.5-1)(1.5-1)f[1,1,1,2].$$

Now we calculate the all needed order of divided differences of the functions as follows:

$$f(1) = \frac{2}{1} = 2, \quad f'(1) = -\frac{2}{1^2} = -2, \quad f''(1) = \frac{4}{1^3} = 4,$$

and the value of $f[1,1,1,2]$ can be calculated as follows:

$$\begin{aligned} f[1,1,1,2] &= \frac{f[1,1,2]-f[1,1,1]}{2-1} = f[1,1,2]-f[1,1,1] \\ &= \frac{f[1,2]-f[1,1]}{2-1} - \frac{f''(1)}{2!} \\ &= \frac{f(2)-f(1)}{2-1} - \frac{f'(1)}{1!} - \frac{f''(1)}{2!} \\ &= f(2)-f(1)-f'(1)-\frac{f''(1)}{2}. \end{aligned}$$

Since $f(x) = \dfrac{2}{x}$, so we have, $f'(x) = -\dfrac{2}{x^2}$ and $f''(x) = \dfrac{4}{x^3}$. Thus

$$f[1,1,1,2] = f(2)-f(1)-f'(1)-\frac{f''(1)}{2} = 1-2+2-2 = -1.$$

So

$$p_3(1.5) = f(1) + (1.5-1)(-2) + (1.5-1)(1.5-1)\frac{4}{2} + (1.5-1)(1.5-1)(1.5-1)(-1),$$

$$f(1.5) \approx p_3(1.5) = 2 + (0.5)(-2)) + (0.25)(2) + (0.125)(-1) = 1.3750,$$

the required approximation of $f(1.5)$ and

$$|f(1.5) - p_3(1.5)| = |1.3333 - 1.3750| = 0.0417,$$

the possible absolute error in the approximation. •

Question 6: (5)

Let $f(x) = x^5 + 1$ be defined in the interval $[0.1, 0.2]$. Use the error formula of three-point formula for the approximation of $f''(0.15)$ to find a value of the unknown point η.

Solution. Since the error formula of the three-point central-difference formula for $f''(0.15)$ is

$$E = Exact - Appxox = -\frac{h^2}{12} f^{(4)}(\eta),$$

for some unknown point $\eta \in (0.1, 0.2)$.
Since the exact value of the second derivative of the function at $x_1 = 0.15$ is

$$f''(0.15) = 20(0.15)^3 = 0.0675,$$

and the approximate value of $f''(0.15)$ using three point formula is

$$f''(0.15) \approx \frac{f(0.2) - 2f(0.15) + f(0.1)}{(0.05)^2} = 0.0710,$$

so error E can be calculated as

$$E = 0.0675 - 0.0710 = -0.0038.$$

Using the error formula and $f^(4)(\eta) = 120\eta$, we have

$$-0.0038 = -\frac{(0.05)^2}{12} 120\eta,$$

and solving for η, we get $\eta = 0.1520$. •

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Question 7: (5)

Compute the approximation of the integral $I(f) = \int_1^2 \frac{e^{-x}}{x} dx$ when $h = 0.2$ using the best integration rule. Compute the error bound.

Given $h = 0.2$, and we have $n = \frac{2}{-}10.2 = 5$. So best rule is Trapezoidal rule. The composite Trapezoidal rule for six points can be written as

$$\int_1^2 f(x)\, dx \approx T_5(f) = \frac{0.2}{2}\Big[f(1) + 2\Big(f(1.2) + f(1.4) + f(1.6) + f(1.8)\Big) + f(2)\Big],$$

and by using the given values, we get

$$\int_1^2 \frac{e^{-x}}{x} dx \approx 0.1\Big[1.7259\Big] = 0.1726.$$

The second derivative of the function $f(x) = \frac{e^{-x}}{x}$ can be obtain as

$$f'(x) = -\frac{e^{-x}}{x}\left[1 + \frac{1}{x}\right] \quad \text{and} \quad f''(x) = \frac{e^{-x}}{x}\left[1 + \frac{2}{x} + \frac{2}{x^2}\right].$$

Since $\eta(x)$ is unknown point in $(1, 2)$, therefore, the bound $|f''|$ on $[1, 2]$ is

$$M = \max_{1 \le x \le 2} |f''(x)| = \max_{1 \le x \le 2} \left| \frac{e^{-x}}{x}\left[1 + \frac{2}{x} + \frac{2}{x^2}\right] \right| = 5/e,$$

at $x = 1$. Thus the error formula becomes

$$|E_{T_5}(f)| \le \frac{(0.2)^2(1)}{12}(5/e) = 0.0061,$$

which is the possible maximum error in our approximation. •

Question 8: (5)

Show that second order Taylor's method for the given initial-value problem

$$xy' + xy = x^3, \qquad y(0) = 0.5, \quad n = 2,$$

is

$$y(x_{i+1}) \approx y_{i+1} = y_i[1 - h + \frac{h^2}{2}] + x_i[h + h^2 - \frac{h^2}{2}x_i], \quad i = 0, 1, \ldots, n-1.$$

Use it to find approximation of $y(0.4)$. Compare your approximate solution with the exact solution $y(x) = y(x) = -1.5e^{-x} + x^2 - 2x + 2$.

Solution. Using $f(x,y) = x^2 - y,\ f'(x,y) = 2x - y' = 2x - x^2 + y$, then the Taylor's method of order 2 gets the form

$$y_{i+1} = y_i + h[x_i^2 - y_i] + \frac{h^2}{2}[2x_i - x_i^2 + y_i],$$

and after simplifying, we get

$$y(x_{i+1}) \approx y_{i+1} = y_i[1 - h + \frac{h^2}{2}] + x_i[hx_i + h^2 - \frac{h^2}{2}x_i].$$

Now by taking $i = 0$ in the above formula, we obtain

$$y(x_{i+1}) \approx y_{i+1} = y_i[1 - h + \frac{h^2}{2}] + x_i[h + h^2 - \frac{h^2}{2}x_i],$$

by using $x_0 = 0, y_0 = 0.5$ and $h = 0.2$, we obtain,
$y(0.2) \approx y_1 = 0.39$ and $y(0.4) \approx y_2 = 0.3194$. Also,

$$Error = |0.35452 - 0.3194| = 0.03512,$$

the required absolute error. •