King Saud University
College of Sciences
Department of Mathematics

M-104
GENERAL MATHEMATICS -2-

CLASS NOTES
DRAFT - 2016

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CHAPTER ONE

CONIC SECTIONS

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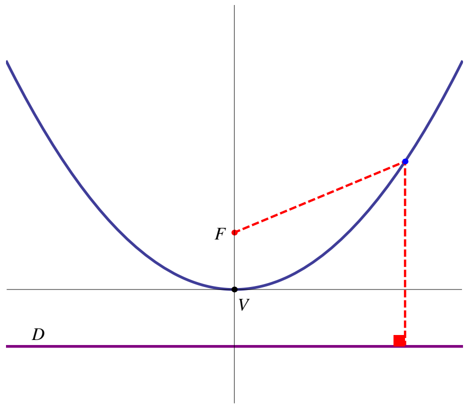
1. Parabola

Definition: A **parabola** is the set of all points in the plane equidistant from a fixed point F (called the **focus**) and a fixed line D (called the **directrix**) in the same plane.

Notes:

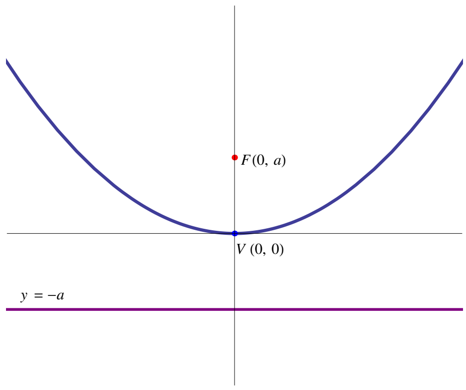
1. The line passing through the focus F and perpendicular to the directrix D is called the **axis** of the parabola .

2. The point half-way from the focus F to the directrix D is called the **vertex** of the parabola and is denoted by V .



1.1 The vertex of the parabola is the origin :
This section discusses the special case where the vertex of the parabola is $(0, 0)$. There are four different cases :

1) $x^2 = 4ay$, where $a > 0$



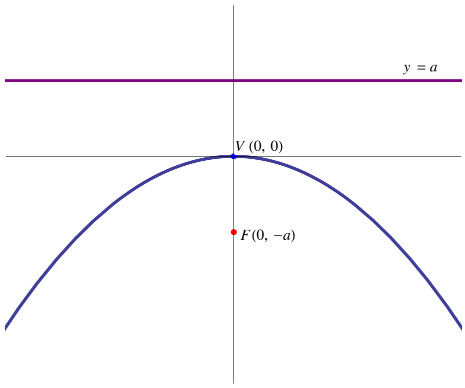
The parabola opens upwards .

The focus is $F(0, a)$.

The equation of the directrix is $y = -a$.

The axis of the parabola is the y-axis .

2) $x^2 = -4ay$, where $a > 0$



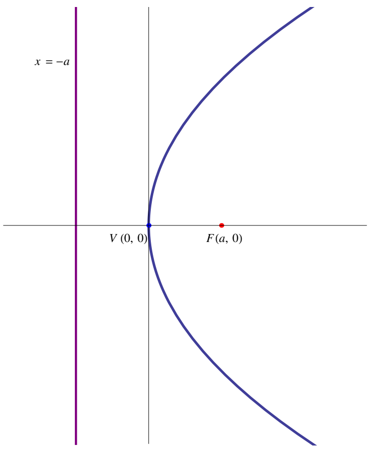
The parabola opens downwards (note the negative sign in the formula).

The focus is $F(0, -a)$.

The equation of the directrix is $y = a$.

The axis of the parabola is the y-axis .

3) $y^2 = 4ax$, where $a > 0$



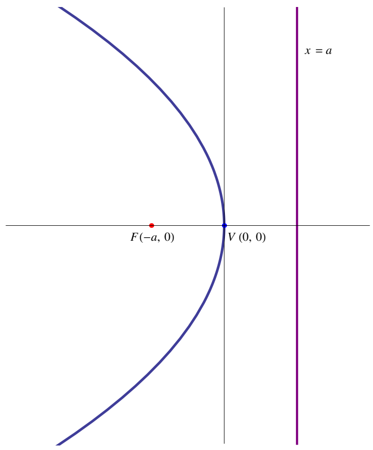
The parabola opens to the right.

The focus is $F(a, 0)$.

The equation of the directrix is $x = -a$.

The axis of the parabola is the x-axis .

4) $y^2 = -4ax$, where $a > 0$



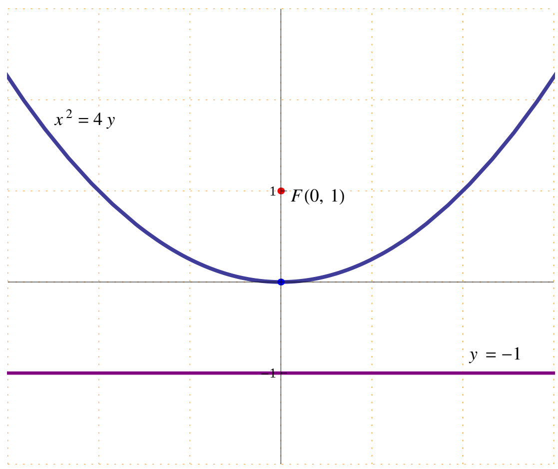
The parabola opens to the left (note the negative sign in the formula) .

The focus is $F(-a, 0)$.

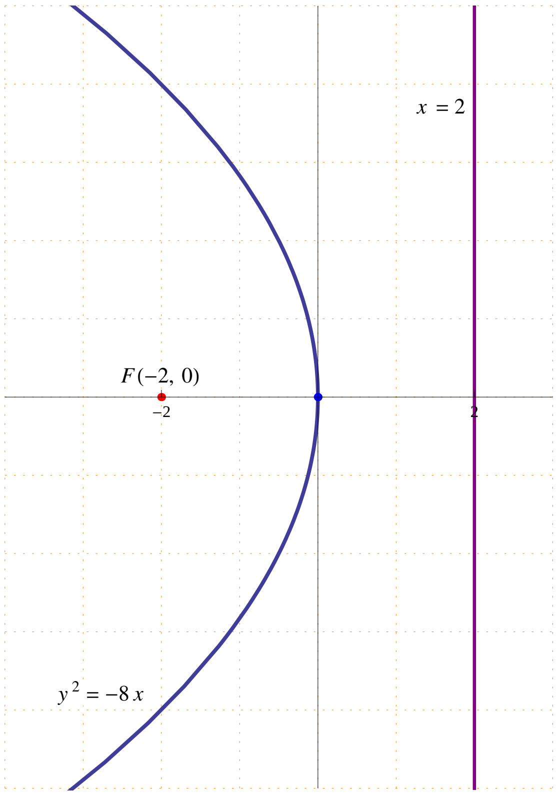
The equation of the directrix is $x = a$.

The axis of the parabola is the x-axis .

Example 1: Find the focus and the directrix of the parabola $x^2 = 4y$, and sketch its graph.
Solution: Since the variable x is of degree 2 and the formula contains a positive sign then $x^2 = 4y$ is similar to case(1), where the parabola opens upwards .
$4a = 4 \;\Rightarrow\; a = 1$
The focus is F(0,1) , and the equation of the directrix is $y = -1$.



Example 2: Find the focus and the directrix of the parabola $y^2 = -8x$, and sketch its graph.
Solution: Since the variable y is of degree 2 and the formula contains a negative sign then $y^2 = -8x$ is similar to case(4), where the parabola opens to the left .
$-4a = -8 \Rightarrow a = 2$
The focus is F(-2,0) , and the equation of the directrix is $x = 2$.

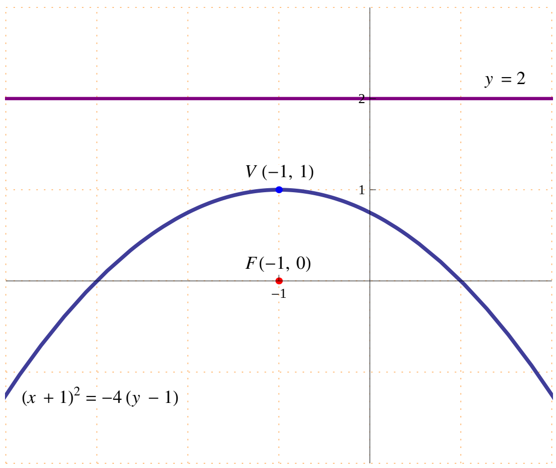


1.2 The general formula of a parabola :

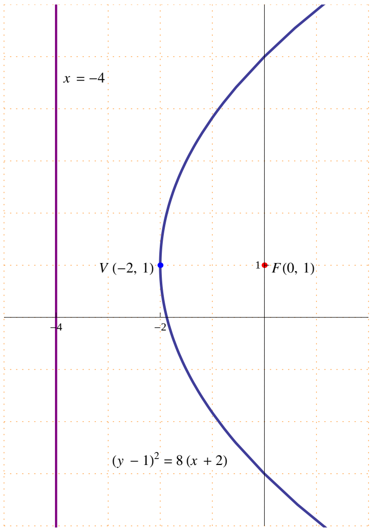
This section discusses the general formula of a parabol where the vertex of the parabola is any point $V(h,k)$ in the plane.
There are four different cases :

No.	The general formula	Focus	Directrix	The parabola opens
1	$(x-h)^2 = 4a(y-k)$	$F(h,k+a)$	$y=k-a$	upwards
2	$(x-h)^2 = -4a(y-k)$	$F(h,k-a)$	$y=k+a$	downwards
3	$(y-k)^2 = 4a(x-h)$	$F(h+a,k)$	$x=h-a$	to the right
4	$(y-k)^2 = -4a(x-h)$	$F(h-a,k)$	$x=h+a$	to the left

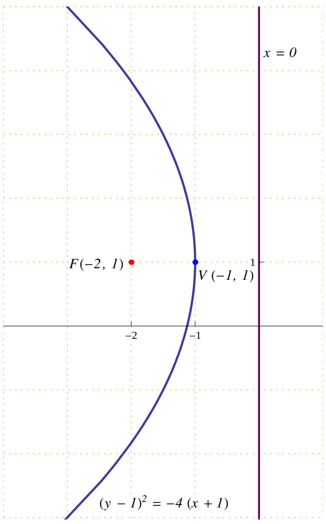
Example 1: Find the focus and the directrix of the parabola $(x+1)^2 = -4(y-1)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (2).
$(x-h)^2 = (x+1)^2 = (x-(-1))^2 \;\Rightarrow\; h=-1$.
$(y-k) = (y-1) \;\Rightarrow\; k=1$.
$-4a = -4 \;\Rightarrow\; a=1$.
The vertex is $V(-1,1)$
The focus is $F(-1,0)$ and the equation of the directrix is $y=2$.
The parabola opens downwards (note the negative sign in the formula).



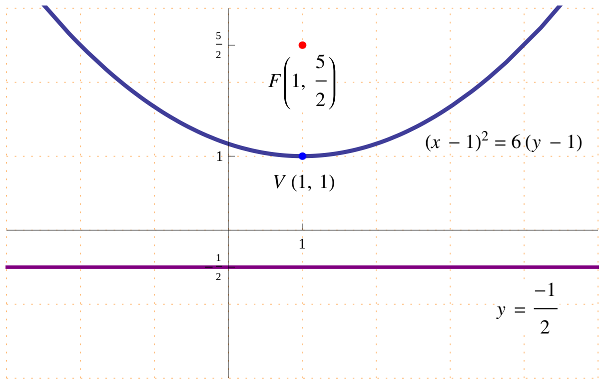
Example 2: Find the focus and the directrix of the parabola $(y-1)^2 = 8(x+2)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (3).
$(y-k)^2 = (y-1)^2 \;\Rightarrow\; k=1$.
$(x-h) = (x+2) = (x-(-2)) \;\Rightarrow\; h=-2$.
$4a = 8 \;\Rightarrow\; a=2$.
The vertex is $V(-2,1)$
The focus is $F(0,1)$ and the equation of the directrix is $x=-4$.
The parabola opens to the right .



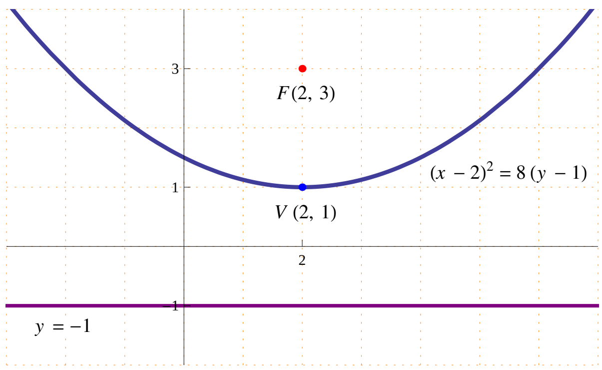
Example 3: Find the focus and the directrix of the parabola $2y^2-4y+8x+10 = 0$, and sketch its graph.
Solution : By completing the square
$2y^2-4y+8x+10=0 \;\Rightarrow\; 2y^2-4y=-8x-10 \;\Rightarrow\; 2(y^2-2y)=-8x-10$
$\Rightarrow\; 2(y^2-2y+1)=-8x-10+2 \;\Rightarrow\; 2(y-1)^2=-8x-8 \Rightarrow\; 2(y-1)^2=-8(x+1)$
$\Rightarrow\; (y-1)^2=-4(x+1)$
The equation of the parabola is similar to case (4).
$(y-k)^2=(y-1)^2 \;\Rightarrow\; k=1$.
$(x-h)=(x+1)=(x-(-1)) \;\Rightarrow\; h=-1$.
$-4a=-4 \;\Rightarrow\; a=1$.
The vertex is $V(-1,1)$.
The focus is $F(-2,1)$ and the equation of the directrix is $x=0$ (the y-axis).
The parabola opens to the left (note the negative sign in the formula)



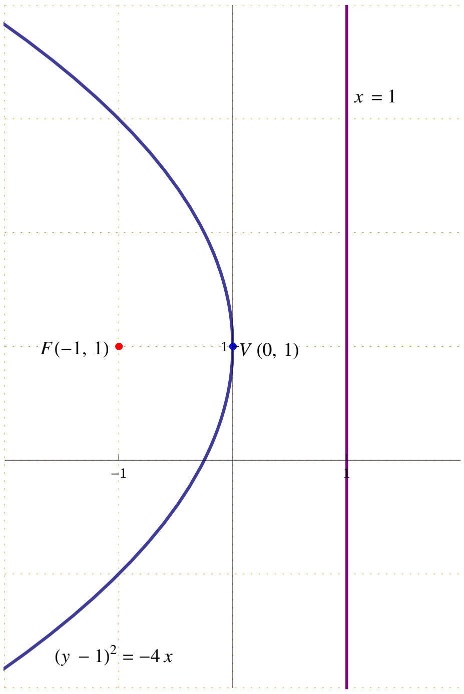
Example 4: Find the focus and the directrix of the parabola $x^2 - 6y - 2x = -7$, and sketch its graph.
Solution : By completing the square
$x^2 - 6y - 2x = -7 \ \Rightarrow \ x^2 - 2x = 6y - 7 \ \Rightarrow \ x^2 - 2x + 1 = 6y - 7 + 1$
$\Rightarrow \ (x-1)^2 = 6y - 6 \ \Rightarrow \ (x-1)^2 = 6(y-1)$
The equation of the parabola is similar to case (1).
$(x-h)^2 = (x-1)^2 \ \Rightarrow \ h = 1$.
$(y-k) = (y-1)) \ \Rightarrow \ k = 1$.
$4a = 6 \ \Rightarrow \ a = \frac{6}{4} = \frac{3}{2}$.
The vertex is $V(1,1)$
The focus is $F\left(1, \frac{5}{2}\right)$ and the equation of the directrix is $y = -\frac{1}{2}$.
The parabola opens upwards.



Example 5: Find the equation of the parabola with vertex $V(2,1)$ and focus $F(2,3)$ and sketch its graph.
Solution : Since the focus is located upper than the vertex then the parabola opens upwards.
Hence its equation is $(x-h)^2 = 4a(y-k)$.
Since the vertex is $V(2,1)$ then $h = 2$ and $k = 1$
a equals the distance between $V(2,1)$ and $F(2,3)$ which equals 2 .
The equation of the parabola with $V(2,1)$ and $F(2,3)$ is $(x-2)^2 = 8(y-1)$



Example 6: Find the equation of the parabola with focus $F(-1,1)$ and directrix $x=1$ and sketch its graph.
Solution : Since the focus is located to the left of the directrix then the parabola opens to the left.
Hence its equation is $(y-k)^2 = -4a(x-h)$.
The vertex is half-way beween the focus and the directrix , hence $V(0,1)$
a equals the distance between $V(0,1)$ and $F(-1,1)$ which equals 1 .
The equation of the parabola with $F(-1,1)$ and directrix $x=1$ is $(y-1)^2 = -4x$

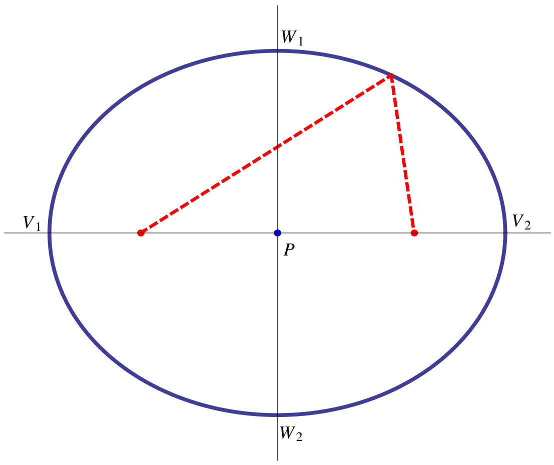


2. Ellipse

Definition: An **ellipse** is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the ellipse and are denoted by F_1 and F_2.
2. The midpoint between F_1 and F_2 is called the **center** of the ellipse and is denoted by P.
3. The endpoints of the **major axis** are called the vertices of the ellipse and are denoted by V_1 and V_2.
4. The endpoints of the **minor axis** are denoted by W_1 and W_2.



2.1 The center of the ellipse is the origin :
This section discusses the special case where the center of the ellipse is $(0,0)$. There are two different cases :

1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$:

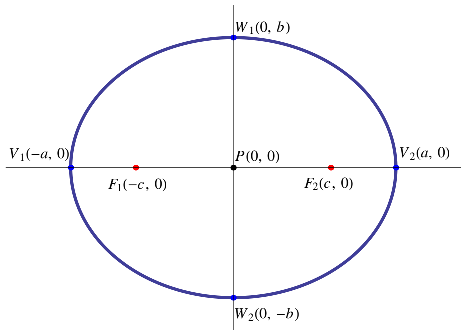
The foci of the ellipse are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 - b^2}$.

The vertices of the ellipse are $V_1(-a, 0)$ and $V_2(a, 0)$.

The endpoints of the minor axis are $W_1(0, b)$ and $W_2(0, -b)$.

The major axis lies on the x-axis , and its length is $2a$.

The minor axis lies on the y-axis , and its length is $2b$.



2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a$:

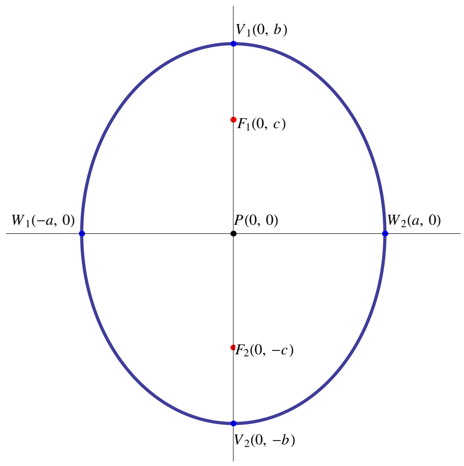
The foci of the ellipse are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{b^2 - a^2}$.

The vertices of the ellipse are $V_1(0, b)$ and $V_2(0, -b)$.

The endpoints of the minor axis are $W_1(-a, 0)$ and $W_2(a, 0)$.

The major axis lies on the y-axis , and its length is $2b$.

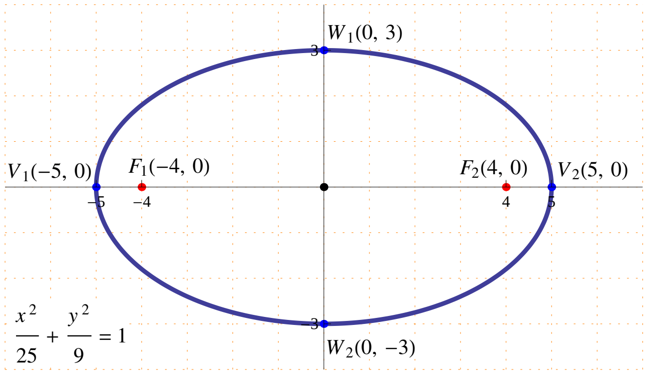
The minor axis lies on the x-axis , and its length is $2a$.



Example 1: Identify the features of the ellipse $9x^2 + 25y^2 = 225$, and sketch its graph.

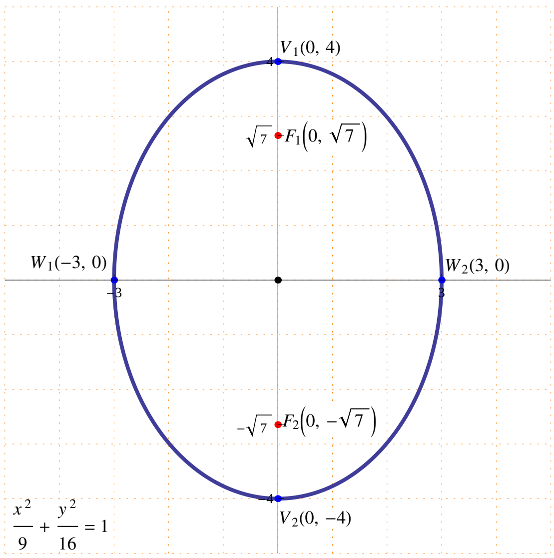
Solution : $9x^2 + 25y^2 = 225 \;\Rightarrow\; \dfrac{9x^2}{225} + \dfrac{25y^2}{225} = 1 \;\Rightarrow\; \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$
$a^2 = 25 \;\Rightarrow\; a = 5$ and $b^2 = 9 \;\Rightarrow\; b = 3$.
Since $a > b$ then $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ is similar to case (1).
$c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4$.
The foci are $F_1\,(-4, 0)$ and $F_2\,(4, 0)$.
The vertices are $V_1(-5, 0)$ and $V_2(5, 0)$.

The endpoints of the minor axis are $W_1(0,3)$ and $W_2(0,-3)$.
The length of the major axis is $2a = 10$.
The length of the minor axis is $2b = 6$.



Example 2: Identify the features of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

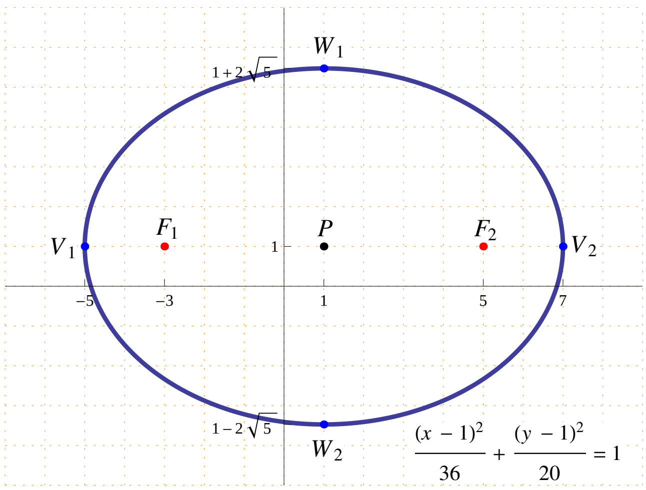
Solution : $16x^2 + 9y^2 = 144 \;\Rightarrow\; \dfrac{16x^2}{144} + \dfrac{9y^2}{144} = 1 \;\Rightarrow\; \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$
$a^2 = 9 \;\Rightarrow\; a = 3$ and $b^2 = 16 \;\Rightarrow\; b = 4$.
Since $b > a$ then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ is similar to case (2).
$c^2 = \sqrt{b^2 - a^2} = \sqrt{16 - 9} = \sqrt{7}$.
The foci are $F_1\left(0, \sqrt{7}\right)$ and $F_2\left(0, -\sqrt{7}\right)$.
The vertices are $V_1(0,4)$ and $V_2(0,-4)$.
The endpoints of the minor axis are $W_1(-3,0)$ and $W_2(3,0)$.
The length of the major axis is $2b = 8$.
The length of the minor axis is $2a = 6$.



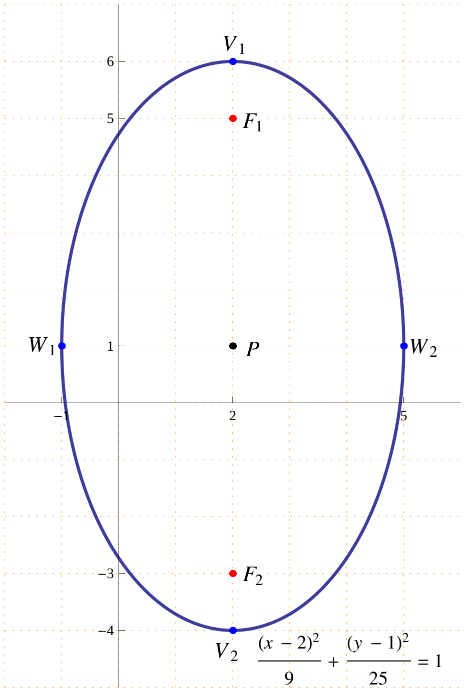
2.2 The general formula of an ellipse :
This section discusses the general formula of an ellipse where the center of the ellipse is any point $P(h,k)$ in the plane.
There are two different cases :

No.	The general Formula	The Foci	The Vertices	W_1 and W_2
1	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $(\ a>b\ )$ and $c=\sqrt{a^2-b^2}$	$F_1(h-c,k)$ $F_2(h+c,k)$	$V_1(h-a,k)$ $V_2(h+a,k)$	$W_1(h,k-b)$ $W_2(h,k+b)$
2	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $(\ b>a\ )$ and $c=\sqrt{b^2-a^2}$	$F_1(h,k-c)$ $F_2(h,k+c)$	$V_1(h,k-b)$ $V_2(h,k+b)$	$W_1(h-a,k)$ $W_2(h+a,k)$

Example 1: Find the equation of the ellipse with foci at $(-3,1)$, $(5,1)$, and one of its vertices is $(7,1)$, and sketch its graph.
Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two foci, hence $(h,k)=\left(\frac{-3+5}{2},\frac{1+1}{2}\right)=(1,1)$.
c is the distance between the center and one of the foci , and it equals to 4 (see the figure).
Since the major axis (where the two foci lie) is parallel to the x-axis , then the general formula of the ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$.
a is the distance between the center and one of the vertices, and it equals 6 (see the figure).
$c^2=a^2-b^2 \ \Rightarrow\ (4)^2=(6)^2-b^2 \ \Rightarrow\ b^2=36-16=20 \ \Rightarrow\ b=2\sqrt{5}$.
The equation of the ellipse is $\frac{(x-1)^2}{36}+\frac{(y-1)^2}{20}=1$.
The vertices of the ellipse are $V_1(-5,1)$ and $V_2(7,1)$.
The endpoints of the minor axis are $W_1\left(1,1+2\sqrt{5}\right)$ and $W_2\left(1,1-2\sqrt{5}\right)$.

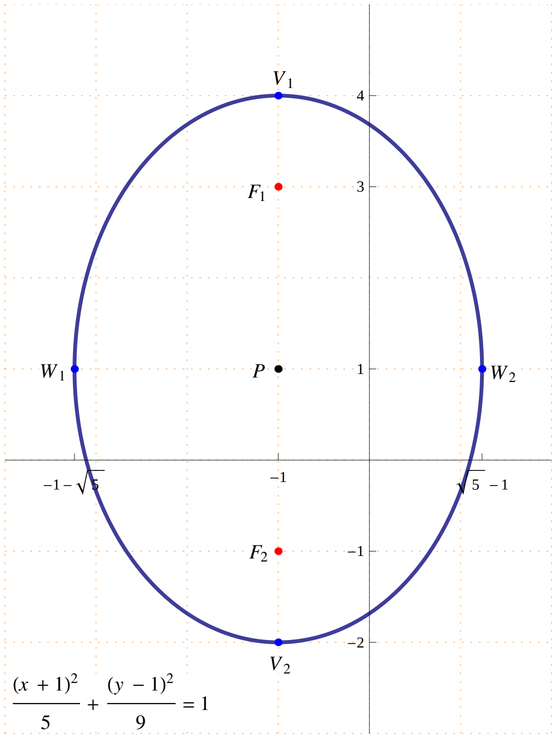


Example 2: Find the equation of the ellipse with foci at $(2,5)$, $(2,-3)$, and the length of its minor axis equals 6 , and sketch its graph.
Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two foci, hence $(h,k) = \left(\dfrac{2+2}{2}, \dfrac{-3+5}{2}\right) = (2,1)$.
c is the distance between the center and one of the foci , and it equals to 4 (see the figure).
Since the major axis (where the two foci lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.
The length of the minor axis is 6 means that $2a = 6 \;\Rightarrow\; a = 3$.
$c^2 = b^2 - a^2 \;\Rightarrow\; (4)^2 = b^2 - (3)^2 \;\Rightarrow\; b^2 = 16 + 9 = 25 \;\Rightarrow\; b = 5$.
The equation of the ellipse is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{25} = 1$.
The vertices of the ellipse are $V_1(2,6)$ and $V_2(2,-4)$.
The endpoints of the minor axis are $W_1(-1,1)$ and $W_2(5,1)$.

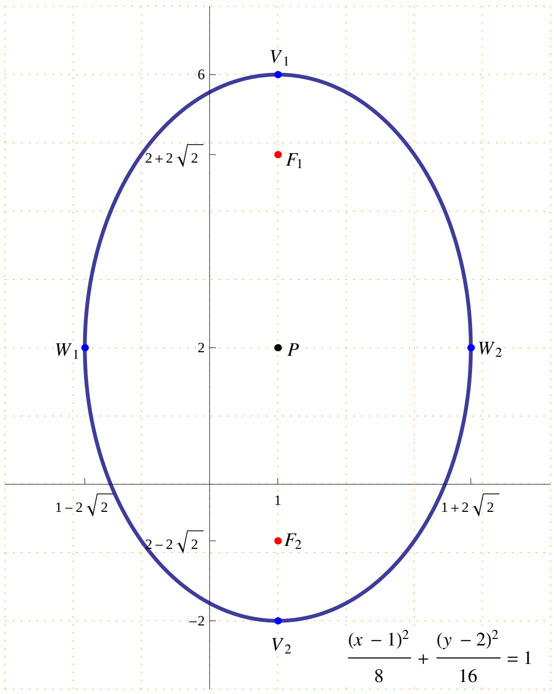


Example 3: Find the equation of the ellipse with vertices at $(-1,4)$, $(-1,-2)$ and the distance between its two foci equals 4 , and sketch its graph.
Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two vertices, hence $(h,k) = \left(\dfrac{-1-1}{2}, \dfrac{-2+4}{2}\right) = (-1,1)$.
The distance between the two foci equals 4 means that $2c = 4 \;\Rightarrow\; c = 2$.
Since the major axis (where the two vertices lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.
The length of the major axis (the distance between the two vertices) equals 6 ,

this means $2b = 6 \;\Rightarrow\; b = 3$.
$c^2 = b^2 - a^2 \;\Rightarrow\; (2)^2 = (3)^2 - a^2 \;\Rightarrow\; a^2 = 9 - 4 = 5 \;\Rightarrow\; a = \sqrt{5}$.
The equation of the ellipse is $\dfrac{(x+1)^2}{5} + \dfrac{(y-1)^2}{9} = 1$.
The foci of the ellipse are $F_1(-1, 3)$ and $F_2(-1, -1)$.
The endpoints of the minor axis are $W_1\left(-1 - \sqrt{5}, 1\right)$ and $W_2\left(-1 + \sqrt{5}, 1\right)$.



Example 4: Identify the features of the ellipse $4x^2 + 2y^2 - 8x - 8y - 20 = 0$, and sketch its graph.
Solution :
$4x^2 + 2y^2 - 8x - 8y - 20 = 0 \;\Rightarrow\; (4x^2 - 8x) + (2y^2 - 8y) = 20$
$\Rightarrow\; 4(x^2 - 2x) + 2(y^2 - 4y) = 20$
By completing the square
$4(x^2 - 2x) + 2(y^2 - 4y) = 20 \;\Rightarrow\; 4(x^2 - 2x + 1) + 2(y^2 - 4y + 4) = 20 + 12$
$\Rightarrow\; 4(x-1)^2 + 2(y-2)^2 = 32$
$\Rightarrow\; \dfrac{4(x-1)^2}{32} + \dfrac{2(y-2)^2}{32} = 1 \;\Rightarrow\; \dfrac{(x-1)^2}{8} + \dfrac{(y-2)^2}{16} = 1$
$b^2 = 16 \;\Rightarrow\; b = 4$ and $a^2 = 8 \;\Rightarrow\; b = \sqrt{8} = 2\sqrt{2}$.
$c^2 = b^2 - a^2 \;\Rightarrow\; c^2 = 16 - 8 = 8 \;\Rightarrow\; c = \sqrt{8} = 2\sqrt{2}$.
The center of the ellipse is $(1, 2)$.
The foci of the ellipse are $F_1\left(1, 2 + 2\sqrt{2}\right)$ and $F_2\left(1, 2 - 2\sqrt{2}\right)$.
The vertices of the ellipse are $V_1(1, 6)$ and $V_2(1, -2)$.
The endpoints of the minor axis are $W_1\left(1 - 2\sqrt{2}, 2\right)$ and $W_2\left(1 + 2\sqrt{2}, 2\right)$
The length of the major axis is 8 and the length of the minor axis is $2\sqrt{8} = 4\sqrt{2}$.



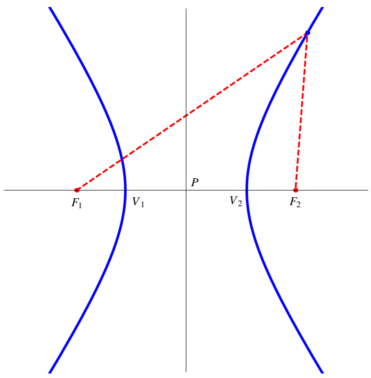
V_1
6
$2+2\sqrt{2}$
F_1
W_1
2
P
W_2
$1-2\sqrt{2}$
1
$1+2\sqrt{2}$
$2-2\sqrt{2}$
F_2
-2
V_2
$\frac{(x-1)^2}{8}+\frac{(y-2)^2}{16}=1$

3. Hyperbola

Definition: A **hyperbola** is the set of all points in the plane for which the difference of the distances between two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the hyperbola and are denoted by F_1 and F_2.
2. The midpoint between F_1 and F_2 is called the **center** of the hyperbola and is denoted by P.



3.1 The center of the hyperbola is the origin :
This section discusses the special case where the center of the hyperbola is $(0, 0)$. There are two different cases :

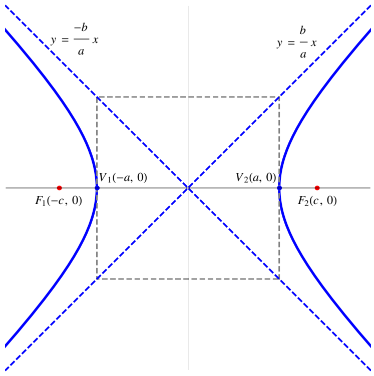
1) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$:

   The foci of the hyperbola are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 + b^2}$.

   The vertices of the hyperbola are $V_1(-a, 0)$ and $V_2(a, 0)$.

   The line segment between V_1 and V_2 is the **transverse axis** , it lies on the x-axis and its length is $2a$.

   The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.



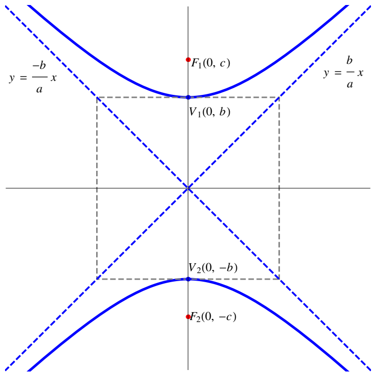
2) $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, where $a > 0$ and $b > 0$:

The foci of the hyperbola are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{a^2 + b^2}$.

The vertices of the hyperbola are $V_1(0, b)$ and $V_2(0, -b)$.

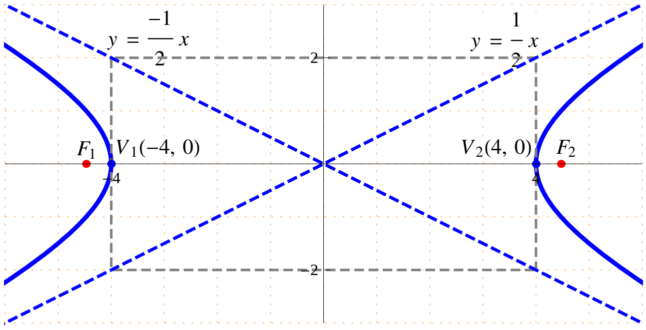
The line segment between V_1 and V_2 is the **transverse axis** , it lies on the y-axis and its length is $2b$.

The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.

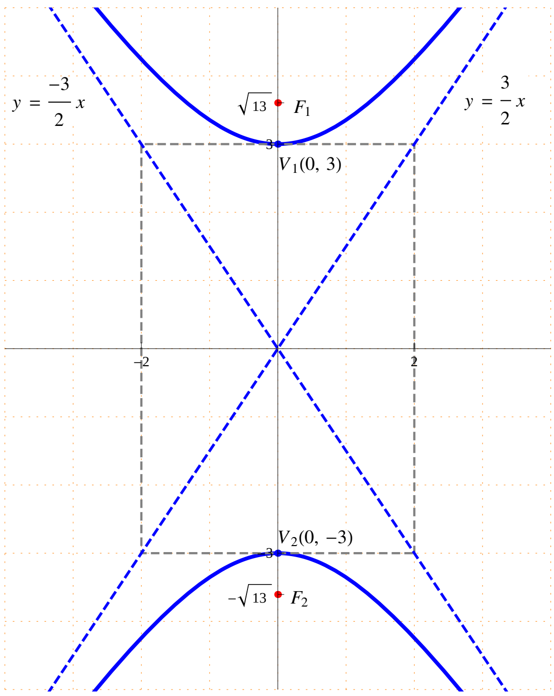


Example 1: Identify the features of the hyperbola $4x^2 - 16y^2 = 64$, and sketch its graph.
Solution :
$4x^2 - 16y^2 = 64 \;\Rightarrow\; \dfrac{4x^2}{64} - \dfrac{16y^2}{64} = 1 \;\Rightarrow\; \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$
This form is similar to case (1).
$a^2 = 16 \;\Rightarrow\; a = 4$ and $b^2 = 4 \;\Rightarrow\; b = 2$
$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$
The foci of the hyperbola are $F_1\left(-2\sqrt{5}, 0\right)$ and $F_2\left(2\sqrt{5}, 0\right)$.
The vertices are $V_1(-4, 0)$ and $V_2(4, 0)$.
The transverse axis lies on the x-axis and its length is $2a = 8$.

The equations of the asymptotes are $y = \frac{2}{4}x = \frac{1}{2}x$ and $y = -\frac{2}{4}x = -\frac{1}{2}x$



Example 2: Identify the features of the hyperbola $4y^2 - 9x^2 = 36$, and sketch its graph.
Solution :
$4y^2 - 9x^2 = 36 \;\Rightarrow\; \frac{4y^2}{36} - \frac{9x^2}{36} = 1 \;\Rightarrow\; \frac{y^2}{9} - \frac{x^2}{4} = 1$
This form is similar to case (2).
$a^2 = 4 \;\Rightarrow\; a = 2$ and $b^2 = 9 \;\Rightarrow\; b = 3$
$c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$
The foci of the hyperbola are $F_1\left(0, \sqrt{13}\right)$ and $F_2\left(0, -\sqrt{13}, \right)$.
The vertices are $V_1(0, 3)$ and $V_2(0, -3)$.
The transverse axis lies on the y-axis and its length is $2b = 6$.
The equations of the asymptotes are $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$



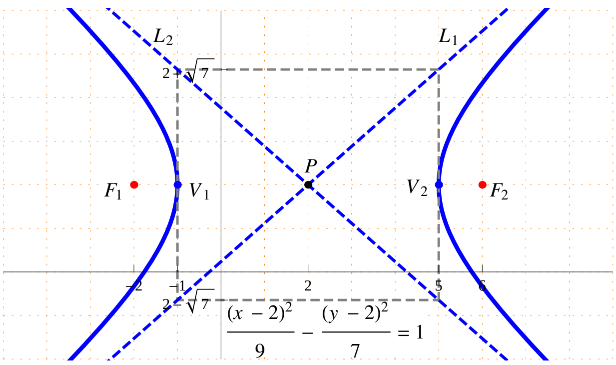
3.2 The general formula of a hyperbola :

This section discusses the general formula of a hyperbola where the center of the hyperbola is any point $P(h, k)$ in the plane.
There are two different cases :

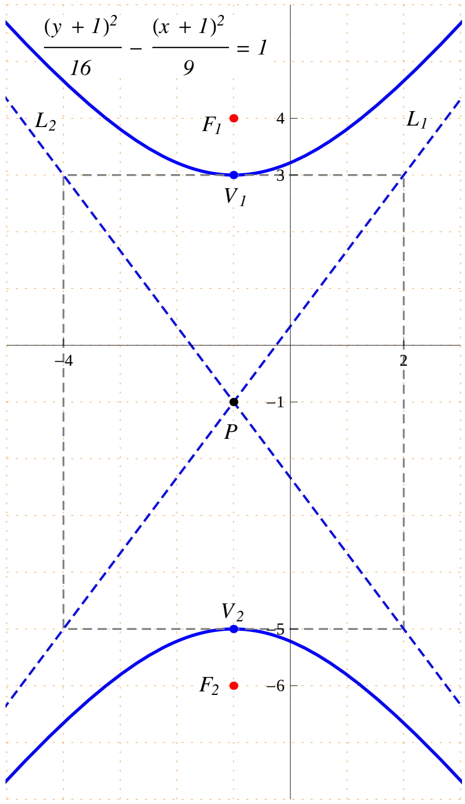
No.	The general Formula	The Foci	The Vertices	Transverse axis
1	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ $(\ c^2 = a^2 + b^2\ )$	$F_1(h-c, k)$ $F_2(h+c, k)$	$V_1(h-a, k)$ $V_2(h+a, k)$	parallel to the x-axis
2	$\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$ $(\ c^2 = a^2 + b^2\ )$	$F_1(h, k+c)$ $F_2(h, k-c))$	$V_1(h, k+b)$ $V_2(h, k-b)$	parallel to the y-axis

The equations of the asymptotes are $y = \dfrac{b}{a}(x-h) + k$ and $y = -\dfrac{b}{a}(x-h) + k$

Example 1: Find the equation of the hyperbola with foci at $(-2, 2)$, $(6, 2)$ and one of its vertices is $(5, 2)$, and sketch its graph.
Solution :
The center of the hyperbola $P(h, k)$ is located in the middle of the two foci , hence $(h, k) = \left(\dfrac{-2+6}{2}, \dfrac{2+2}{2}\right) = (2, 2)$
Note that the two foci lie on a line parallel to the x-axis , hence the general formula of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.
$2c$ is the distance between the two foci , hence $2c = 8 \ \Rightarrow\ c = 4$.
a is the distance between the center $(2, 2)$ and the vertex $(5, 2)$, hence $a = 3$, and the other vertex is $(-1, 2)$.
$c^2 = a^2 + b^2 \ \Rightarrow\ 4^2 = 3^2 + b^2 \ \Rightarrow\ b^2 = 16 - 9 = 7 \ \Rightarrow\ c = \sqrt{7}$.
The equation of the hyperbola is $\dfrac{(x-2)^2}{9} - \dfrac{(y-2)^2}{7} = 1$
The equations of the asymptotes are $L_1 : y = \dfrac{\sqrt{7}}{3}(x-2) + 2$ and
$L_2 : y = -\dfrac{\sqrt{7}}{3}(x-2) + 2$

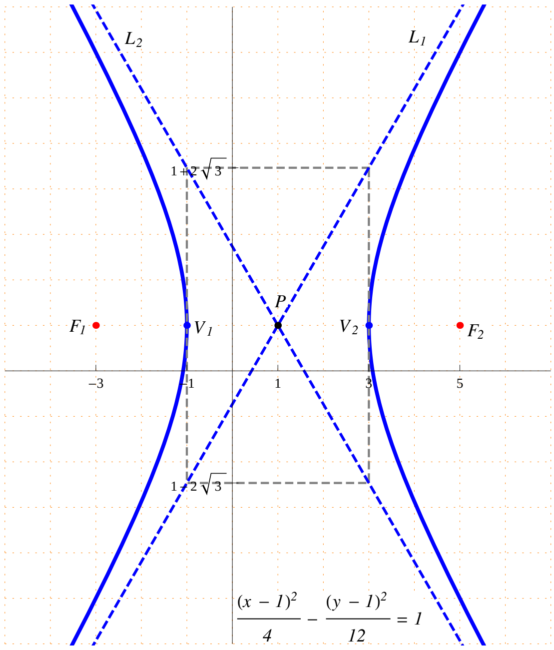


Example 2: Find the equation of the hyperbola with foci at $(-1,-6)$, $(-1,4)$ and the length of its transverse axis is 8 , and sketch its graph.
Solution :
The center of the hyperbola $P(h,k)$ is located in the middle of the two foci , hence $(h,k) = \left(\dfrac{-1-1}{2}, \dfrac{-6+4}{2}\right) = (-1,-1)$
Note that the two foci lie on a line parallel to the y-axis , hence the general formula of the hyperbola is $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$.
$2c$ is the distance between the two foci , hence $2c = 10 \ \Rightarrow \ c = 5$.
The length of the transverse axis is 8 , this means $2b = 8 \ \Rightarrow \ b = 4$.
The vertices are $(-1,-5)$ and $(-1,3)$.
$c^2 = a^2 + b^2 \ \Rightarrow \ 5^2 = a^2 + 4^2 \ \Rightarrow \ a^2 = 25 - 16 = 9 \ \Rightarrow \ a = 3$.
The equation of the hyperbola is $\dfrac{(y+1)^2}{16} - \dfrac{(x+1)^2}{9} = 1$.
The equations of the asymptotes are $L_1 : \ y = \dfrac{4}{3}(x+1) - 1$ and
$L_2 : \ y = -\dfrac{4}{3}(x+1) - 1$



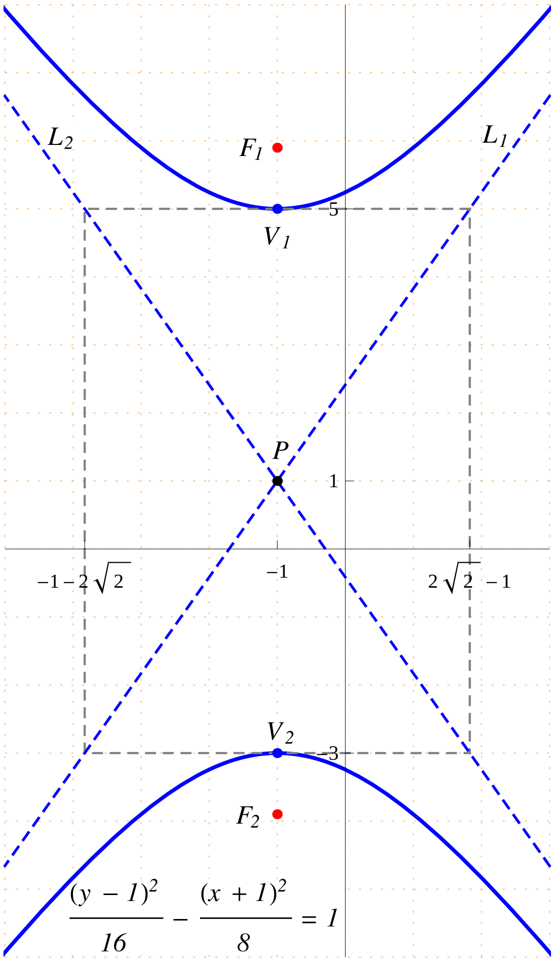
Example 3: Find the equation of the hyperbola with center at $(1,1)$, one of its foci is $(5,1)$ and one of its vertices is $(-1,1)$, and sketch its graph.
Solution :
Since the center and the focus lie on a line parallel to the x-axis , then the

general formula of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.
c is the distance between the center $(1,1)$ and the focus $(5,1)$, hence $c = 4$, the other foci is $(-3,1)$.
a is the distance between the center $(1,1)$ and the vertex $(-1,1)$, hence $a = 2$, the other vertex is $(3,1)$.
$c^2 = a^2 + b^2 \Rightarrow 4^2 = 2^2 + b^2 \Rightarrow b^2 = 16 - 4 = 12 \Rightarrow b = \sqrt{12} = 2\sqrt{3}$
The equation of the hyperbola is $\dfrac{(x-1)^2}{4} - \dfrac{(y-1)^2}{12} = 1$.
The equations of the asymptotes are
$L_1 : \ y = \dfrac{2\sqrt{3}}{2}(x-1) + 1 = \sqrt{3}(x-1) + 1$ and $L_2 : \ y = -\sqrt{3}(x-1) + 1$



Example 4: Identify the features of the hyperbola $2y^2 - 4x^2 - 4y - 8x - 34 = 0$, and sketch its graph.
Solution :
$2y^2 - 4x^2 - 4y - 8x - 34 = 0 \Rightarrow (2y^2 - 4y) - (4x^2 + 8x) = 34$
$\Rightarrow 2(y^2 - 2y) - 4(x^2 + 2x) = 34$
$\Rightarrow 2(y^2 - 2y + 1) - 4(x^2 + 2x + 1) = 34 + 2 - 4 \Rightarrow 2(y-1)^2 - 4(x+1)^2 = 32$
$\Rightarrow \dfrac{2(y-1)^2}{32} - \dfrac{4(x+1)^2}{32} = 1 \Rightarrow \dfrac{(y-1)^2}{16} - \dfrac{(x+1)^2}{8} = 1$
$b^2 = 16 \Rightarrow b = 4$ and $a^2 = 8 \Rightarrow a = \sqrt{8} = 2\sqrt{2}$.
$c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 8 = 24 \Rightarrow c = \sqrt{24} = 2\sqrt{6}$.
The center of the hyperbola is $P(-1,1)$.
The foci of the hyperbola are $F_1\left(-1, 1 + 2\sqrt{6}\right)$ and $F_2\left(-1, 1 - 2\sqrt{6}\right)$.
The vertices of the hyperbola are $V_1(-1,5)$ and $V_2(-1,-3)$.
The transverse axis is parallel to the y-axis and its length is $2b = 8$.
The equations of the asymptotes are

$$L_1: \ y = \frac{4}{2\sqrt{2}}(x+1)+1 = \sqrt{2}(x+1)+1 \text{ and } L_2: \ y = -\sqrt{2}(x+1)+1$$



CHAPTER TWO

MATRICES AND DETERMINANTS

1. Matrices

Definition : A **matrix A** of order $m \times n$ is a set of real numbers arranged in a rectangular array of m rows and n columns. It is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notes :

1. a_{ij} represents the element of the matrix **A** that lies in row i and column j.
2. The matrix **A** can also be written as $\mathbf{A} = (a_{ij})_{m \times n}$.
3. If the number of rows equals the number of columns ($m = n$) then **A** is called a **square** matrix of order n.
4. In a square matrix $\mathbf{A} = (a_{ij})$, the set of elements of the form a_{ii} is called the diagonal of the matrix.

Examples :

1. $\begin{pmatrix} -1 & 4 & 0 \\ 2 & -3 & 7 \end{pmatrix}$ is a matrix of order 2×3.

   $a_{11} = -1$, $a_{12} = 4$, $a_{13} = 0$, $a_{21} = 2$, $a_{22} = -3$ and $a_{23} = 7$.
2. $\begin{pmatrix} 5 & -3 & 2 \\ 0 & 1 & 7 \\ 0 & 8 & 13 \end{pmatrix}$ is a square matrix of order 3.

   The diagonal is the set $\{a_{11}, a_{22}, a_{33}\} = \{5, 1, 13\}$

1.1 Special types of matrices :

1. Row vector : A row vector of order n is a matrix of order $1 \times n$, and it is written as $\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$

Example : $\begin{pmatrix} 2 & 7 & 0 & -1 \end{pmatrix}$ is a row vector of order 4.

2. Column vector : A column vector of order n is a matrix of order $n \times 1$, and it is written as $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Example : $\begin{pmatrix} 8 \\ -1 \\ 2 \end{pmatrix}$ is a column vector of order 3.

3. Null matrix : The matrix $(a_{ij})_{m \times n}$ of order $m \times n$ is called a **null matrix** if $a_{ij} = 0$ for all i and j, and it is denoted by $\mathbf{0}$.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Example : $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a null matrix of order 3×4.

4. Upper triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called an **upper triangular matrix** if $a_{ij} = 0$ for all $i > j$, and it is written as $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$

Example : $\begin{pmatrix} 8 & 5 & -2 & 1 \\ 0 & 3 & 1 & -6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is an upper triangular matrix of order 4.

5. Lower triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **lower triangular matrix** if $a_{ij} = 0$ for all $i < j$, and it is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & -5 & 7 \end{pmatrix}$ is a lower triangular matrix of order 3.

6. Diagonal matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$, and it is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a diagonal matrix of order 3.

7. Identity matrix : The square matrix $I_n = (a_{ij})$ of order n is called an **identity matrix** if $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, and it is written as $I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$

Example : $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3.

1.2 Elementary matrix operations :

1. Addition and subtraction of matrices :

Addition or subtraction of two matrices is defined if the two matricest have the same order.

If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ any two matrices of order $m \times n$ then

1. $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m\times n}$.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

2. $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{m\times n}$.

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \dots & a_{mn} - b_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 1 & -4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 2 & 1 \\ -3 & 7 & -2 \end{pmatrix}$ then

$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+5 & -3+2 & 0+1 \\ 1+(-3) & -4+7 & 6+(-2) \end{pmatrix} = \begin{pmatrix} 7 & -1 & 1 \\ -2 & 3 & 4 \end{pmatrix}$

$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2-5 & -3-2 & 0-1 \\ 1-(-3) & -4-7 & 6-(-2) \end{pmatrix} = \begin{pmatrix} -3 & -5 & -1 \\ 4 & -11 & 8 \end{pmatrix}$

Notes:

1. The addition of matrices is commutative : if $\mathbf{A}$ and $\mathbf{B}$ any two matrices of the same order then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

2. The null matrix is the identity element of addition : if $\mathbf{A}$ is any matrix then $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

2. Multiplying a matrix by a scalar :

If $\mathbf{A} = (a_{ij})$ is a matrix of order $m \times n$ and $c \in \mathbb{R}$ then $c\mathbf{A} = (ca_{ij})$.

$$c\mathbf{A} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -2 & 0 \end{pmatrix}$ then $3\mathbf{A} = \begin{pmatrix} 9 & -3 & 12 \\ 6 & -6 & 0 \end{pmatrix}$

3. Multiplying a row vector by a column vector :
If $\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ is a row vector of order n and

$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a column vector of order n then

$$\mathbf{AB} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Example : If $\mathbf{A} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix} = -4 - 4 + 0 - 5 = -13$$

4. Multiplication of matrices :

1. If $\mathbf{A}$ and $\mathbf{B}$ any two matrices then $\mathbf{AB}$ is defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.
2. If $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{n \times p}$ then $\mathbf{AB} = (c_{ij})_{m \times p}$.

   c_{ij} is calculated by multiplying the i^{th} row of $\mathbf{A}$ by the j^{th} column of $\mathbf{B}$.

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

Example 1 :

1. $\begin{pmatrix} -1 & 3 & 4 \\ -2 & 0 & 5 \end{pmatrix}_{2 \times 3} \begin{pmatrix} 1 & 3 \\ -1 & -2 \\ 4 & 0 \end{pmatrix}_{3 \times 2}$

$$= \begin{pmatrix} (-1 \times 1) + (3 \times -1) + (4 \times 4) & (-1 \times 3) + (3 \times -2) + (4 \times 0) \\ (-2 \times 1) + (0 \times -1) + (5 \times 4) & (-2 \times 3) + (0 \times -2) + (5 \times 0) \end{pmatrix}_{2 \times 2}$$

$$= \begin{pmatrix} -1 - 3 + 16 & -3 - 6 + 0 \\ -2 + 0 + 20 & -6 + 0 + 0 \end{pmatrix}_{2 \times 2} = \begin{pmatrix} 12 & -9 \\ 18 & -6 \end{pmatrix}_{2 \times 2}$$

2. $\begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}_{2 \times 2} \begin{pmatrix} 0 & -3 & 4 \\ -2 & 0 & 1 \end{pmatrix}_{2 \times 3}$

$$= \begin{pmatrix} (3\times 0)+(-1\times -2) & (3\times -3)+(-1\times 0) & (3\times 4)+(-1\times 1) \\ (-2\times 0)+(5\times -2) & (-2\times -3)+(5\times 0) & (-2\times 4)+(5\times 1) \end{pmatrix}_{2\times 3}$$

$$\begin{pmatrix} 0+2 & -9+0 & 12-1 \\ 0-10 & 6+0 & -8+5 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 2 & -9 & 11 \\ -10 & 6 & -3 \end{pmatrix}_{2\times 3}$$

Example 2: Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$

Compute (if possible) : $2\mathbf{BA}$ and $\mathbf{AB}$
Solution : $\mathbf{A}$ is of order 3×3 and $\mathbf{B}$ is of order 3×2

$2\mathbf{BA}$ is not possible because the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$.

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}_{3\times 3} \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}_{3\times 2} = \begin{pmatrix} (1-4+0) & (-1-6+12) \\ (4+10+0) & (-4+15+24) \\ (2+0+0) & (-2+0+4) \end{pmatrix}_{3\times 2}$$

$$\mathbf{AB} = \begin{pmatrix} -3 & 5 \\ 14 & 35 \\ 2 & 2 \end{pmatrix}_{3\times 2}$$

Notes :

1. The identity matrix is the identity element in matrix multiplication :

   If A is a matrix of order $m \times n$ and $\mathbf{I}_n$ is the identity matrix of order n then $\mathbf{A}\ \mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$.

2. Matrix multiplication is not commutative :

   If $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

   $$\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 8 & 5 \end{pmatrix}$$

   $$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

   $\mathbf{AB} \neq \mathbf{BA}$.

3. $\mathbf{AB} = \mathbf{0}$ does not imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

   For example, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \mathbf{0}$

   But $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

1.3 Transpose of a matrix :
If $\mathbf{A} = (a_{ij})_{m\times n}$ then the transpose of $\mathbf{A}$ is $\mathbf{A}^t = (a_{ji})_{n\times m}$.

Example : If $\mathbf{A} = \begin{pmatrix} 4 & 0 & -2 \\ -3 & 5 & 1 \end{pmatrix}$ then $\mathbf{A}^t = \begin{pmatrix} 4 & -3 \\ 0 & 5 \\ -2 & 1 \end{pmatrix}$

Note : The transpose of a lower triangular matrix is an upper triangular matrix , and the transpose of an upper triangular matrix is a lower triangular matrix .

Theorem :
If $\mathbf{A}$ and $\mathbf{B}$ any two matrices and $\lambda \in \mathbb{R}$ then

1. $\left(\mathbf{A}^t\right)^t = \mathbf{A}$.
2. $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.
3. $(\lambda\mathbf{A})^t = \lambda\ \mathbf{A}^t$.
4. $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$.

1.4 Properties of operations on matrices :

1. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ any three matrices of the same order then

   $\mathbf{A}+\mathbf{B}+\mathbf{C} = (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{C})+\mathbf{B}$

2. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m\times n$ and $\mathbf{C}$ a matrix of order $n\times p$ then $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

3. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m\times n$ and $\mathbf{C}$ a matrix of order $p\times m$ then $\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$

4. If $\mathbf{A}$ a matrix of order $m\times n$, $\mathbf{B}$ a matrix of order $n\times p$ and $\mathbf{C}$ a matrix of order $p\times q$ then $\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

2. Determinants

If $\mathbf{A}$ is a square matrix then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$.

2.1 The determinant of a 2×2 matrix :

If $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example :

If $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$ then $|\mathbf{A}| = (5 \times 3) - (2 \times -1) = 15 + 2 = 17$

2.2 The determinant of a 3×3 matrix :

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3.

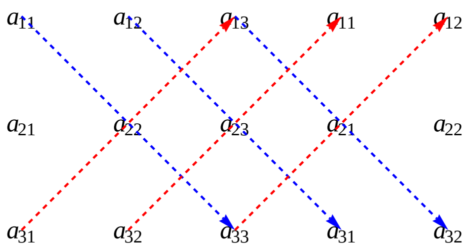
1). The determinant of $\mathbf{A}$ is defined as :

$$|\mathbf{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|\mathbf{A}| = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

2). Sarrus Method for calculating the deteminant of a 3×3 matrix :

Write the first two columns to the right of the matrix to get a 3×5 matrix



$$|\mathbf{A}| = \left(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right) - \left(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}\right)$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 4 \\ -5 & 0 & 1 \end{pmatrix}$

1) Using the definition of the determinant of a 3×3 matrix

$$|\mathbf{A}| = 2 \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 4 \\ -5 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 \\ -5 & 0 \end{vmatrix}$$

$$|\mathbf{A}| = 2\left(2 \times 1 - 4 \times 0\right) - 3\left(1 \times 1 - 4 \times -5\right) - 1\left(1 \times 0 - 2 \times -5\right)$$

$$|\mathbf{A}| = 2(2 - 0) - 3(1 + 20) - 1(0 + 10) = 4 - 63 - 10 = -69$$

2) Using Sarrus Method

$$\begin{array}{ccccc} 2 & 3 & -1 & 2 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ -5 & 0 & 1 & -5 & 0 \end{array}$$

$|\mathbf{A}| = (2 \times 2 \times 1 + 3 \times 4 \times -5 + (-1) \times 1 \times 0) - (-5 \times 2 \times -1 + 0 \times 4 \times 2 + 1 \times 1 \times 3)$

$|\mathbf{A}| = (4 - 60 + 0) - (10 + 0 + 3) = -56 - 13 = -69$.

2.3 The determinant of a 4×4 matrix :

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ be a 4×4 matrix , then

$|\mathbf{A}| = a_{11} |\mathbf{A}_1| - a_{12} |\mathbf{A}_2| + a_{13} |\mathbf{A}_3| - a_{14} |\mathbf{A}_4|$

where

$\mathbf{A}_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} \quad , \quad \mathbf{A}_2 = \begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$

$\mathbf{A}_3 = \begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} \quad , \quad \mathbf{A}_4 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -1 & 5 \\ 2 & 1 & -3 & 0 \\ 1 & -2 & -1 & 3 \end{pmatrix}$

$|\mathbf{A}| = (3) |\mathbf{A}_1| - (1) |\mathbf{A}_2| + (-2) |\mathbf{A}_3| - (1) |\mathbf{A}_4|$

where

$\mathbf{A}_1 = \begin{pmatrix} 4 & -1 & 5 \\ 1 & -3 & 0 \\ -2 & -1 & 3 \end{pmatrix} \quad , \quad \mathbf{A}_2 = \begin{pmatrix} 0 & -1 & 5 \\ 2 & -3 & 0 \\ 1 & -1 & 3 \end{pmatrix}$

$\mathbf{A}_3 = \begin{pmatrix} 0 & 4 & 5 \\ 2 & 1 & 0 \\ 1 & -2 & 3 \end{pmatrix} \quad , \quad \mathbf{A}_4 = \begin{pmatrix} 0 & 4 & -1 \\ 2 & 1 & -3 \\ 1 & -2 & -1 \end{pmatrix}$

- To calculate $|\mathbf{A}_1|$

$$\begin{array}{ccccc} 4 & -1 & 5 & 4 & -1 \\ 1 & -3 & 0 & 1 & -3 \\ -2 & -1 & 3 & -2 & -1 \end{array}$$

$|\mathbf{A}_1| = (-36 + 0 - 5) - (30 + 0 - 3) = -36 - 5 - 30 + 3 = -68$

- To calculate $|\mathbf{A}_2|$

$$\begin{array}{ccccc} 0 & -1 & 5 & 0 & -1 \\ 2 & -3 & 0 & 2 & -3 \\ 1 & -1 & 3 & 1 & -1 \end{array}$$

$|\mathbf{A}_2| = (0+0-10)-(-15+0-6) = -10+21 = 11$

- To calculate $|\mathbf{A}_3|$

$$\begin{array}{ccccc} 0 & 4 & 5 & 0 & 4 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & -2 & 3 & 1 & -2 \end{array}$$

$|\mathbf{A}_3| = (0+0-20)-(5+0+24) = -20-29 = -49$

- To calculate $|\mathbf{A}_4|$

$$\begin{array}{ccccc} 0 & 4 & -1 & 0 & 4 \\ 2 & 1 & -3 & 2 & 1 \\ 1 & -2 & -1 & 1 & -2 \end{array}$$

$|\mathbf{A}_4| = (0-12+4)-(-1+0-8) = -8+9 = 1$

$|\mathbf{A}| = (3)\,|\mathbf{A}_1| - (1)\,|\mathbf{A}_2| + (-2)\,|\mathbf{A}_3| - (1)\,|\mathbf{A}_4|$
$|\mathbf{A}| = (3\times -68)-(1\times 11)+(-2\times -49)-(1\times 1)$
$|\mathbf{A}| = -204-11+98-1 = -216+98 = -118$.

2.4 Properties of determinants :

1. If $\mathbf{A}$ is a square matrix that contains a zero row (or a zero column) then $|\mathbf{A}| = 0$.

   Examples :

   $\begin{vmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{vmatrix} = 0$ (the second row R_2 is a zero row)

   $\begin{vmatrix} 3 & -1 & 0 \\ -1 & 5 & 0 \\ 2 & -2 & 0 \end{vmatrix} = 0$ (the third column C_3 is a zero column)

2. If $\mathbf{A}$ is a square matrix that contains two equal rows (or two equal columns) then $|\mathbf{A}| = 0$.

   Examples :

   $\begin{vmatrix} 4 & -5 & 4 \\ 0 & 2 & 0 \\ -3 & 1 & -3 \end{vmatrix} = 0$ (because $C_1 = C_3$).

   $\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$ (because $R_2 = R_3$)

3. If $\mathbf{A}$ is a square matrix that contains a row which is a multiple of another row (or a column which is a multiple of another column) then $|\mathbf{A}| = 0$.

   Examples :

   $$\begin{vmatrix} 2 & 1 & -3 \\ 0 & 5 & 1 \\ 4 & 2 & -6 \end{vmatrix} = 0 \text{ (because } R_3 = 2R_1\text{)}.$$

   $$\begin{vmatrix} -2 & 1 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0 \text{ (because } C_1 = -2C_2\text{)}.$$

4. If $\mathbf{A}$ is a diagonal matrix or an upper triangular matrix or a lower triangular matrix the $|\mathbf{A}|$ is the the product of the elements of the main diagonal.

   Examples :

   $$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2 \times -1 \times 5 = -10 \text{ (Diagonal matrix)}$$

   $$\begin{vmatrix} 1 & 3 & -7 \\ 0 & 5 & 4 \\ 0 & 0 & -3 \end{vmatrix} = 1 \times 5 \times -3 = 15 \text{ (Upper triangular matrix)}$$

   $$\begin{vmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 7 & 2 \end{vmatrix} = 3 \times 1 \times 2 = 6 \text{ (Lower triangular matrix)}$$

5. The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.

6. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by multiplying one of the rows (or columns) of $\mathbf{A}$ by a non-zero constant λ then $|\mathbf{B}| = \lambda|\mathbf{A}|$.

7. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by interchanging two rows (or two columns) of $\mathbf{A}$ then $|\mathbf{B}| = -|\mathbf{A}|$.

   Example :

   $$\begin{vmatrix} 3 & 0 & 4 \\ 6 & -1 & 2 \\ 0 & 0 & 5 \end{vmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} -1 \times \begin{vmatrix} 6 & -1 & 2 \\ 3 & 0 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$

   $$\xrightarrow{C_1 \longleftrightarrow C_2} -1 \times -1 \times \begin{vmatrix} -1 & 6 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix} = -1 \times -1 \times -1 \times 3 \times 5 = -15$$

8. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by mutliplying a row by a non-zero constant and adding the result to another row (or mutliplying a column by a non-zero constant and adding the result to another column) then $|\mathbf{B}| = |\mathbf{A}|$.

   Example :

$$\begin{vmatrix} 5 & 2 & 3 \\ 15 & 8 & 1 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-3R_1+R_2} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-2R_1+R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 2 & -4 \end{vmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{vmatrix} = 5 \times 2 \times 4 = 40$$

Examples : Use properties of determinants to calculate the derminants of the following matrices

1. $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = 0 \text{ (because } C_3 = \frac{3}{2}C_2)$$

2. $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix}$$

$$\xrightarrow{-R_1+R_3} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} = 0 \text{ (because } R_2 = -8R_3)$$

3. $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix}$$

$$\xrightarrow{-R_3+R_4} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 \end{vmatrix} = 0 \text{ (because } R_2 = R_4)$$

CHAPTER THREE

SYSTEMS OF LINEAR EQUATIONS

1. **Cramer's Rule**

2. **Gauss Elimination Method**

3. **Gauss-Jordan Method**

Systems of Linear Equations

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \qquad (*)$$

Using multiplication of matrices , the above system of linear equations can be written as : $\mathbf{A}\ \mathbf{X} = \mathbf{B}$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

A is called the coefficients matrix
X is called the column vector of variables (or column vector of the unknowns)
B is called the column vector of constants (or column vector of the resultants)

Theorem : The system of linear equations (*) has a solution if $\det(\mathbf{A}) \neq 0$.

This chapter presents three metods of solving the system of linear equations (*), the first method is Cramer's rule , the second is Gauss elimination method , and the third is Gauss-Jordan method .

1. Cramer's rule

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \qquad (*)$$

$$\mathbf{A}\ \mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

If $\det(\mathbf{A}) \neq 0$ then the solution of the system (*) is given by
$x_i = \dfrac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$ for every $i = 1, 2, \cdots, n$.
Where $\mathbf{A}_i$ is the matrix formed by replacing the i^{th} column of $\mathbf{A}$ by the column vector of constants.

$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{pmatrix}$

$\mathbf{A}_n = \begin{pmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{pmatrix}$

Example 1: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{ccccc} 2x & + & 3y & = & 7 \\ -x & + & y & = & 4 \end{array}$$

Solution : In this system of linear equations
$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = (2 \times 1) - (-1 \times 3) = 2 - (-3) = 2 + 3 = 5$

$\mathbf{A}_1 = \begin{pmatrix} 7 & 3 \\ 4 & 1 \end{pmatrix} \implies \det(\mathbf{A}_1) = 7 - 12 = -5$

$\mathbf{A}_2 = \begin{pmatrix} 2 & 7 \\ -1 & 4 \end{pmatrix} \implies \det(\mathbf{A}_2) = 8 - (-7) = 15$

$x = \dfrac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \dfrac{-5}{5} = -1$ and $y = \dfrac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \dfrac{15}{5} = 3$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Example 2: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 3 \\ 4x & + & y & - & z & = & -2 \\ 2x & - & 2y & + & z & = & 6 \end{array}$$

Solution : In this system of linear equations

$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$

To calculate $\det(\mathbf{A})$:

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 4 & 1 & -1 & 4 & 1 \\ 2 & -2 & 1 & 2 & -2 \end{array}$$

$\det(\mathbf{A}) = (2 - 2 - 8) - (2 + 4 + 4) = -8 - 10 = -18$

$\mathbf{A}_1 = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 1 & -1 \\ 6 & -2 & 1 \end{pmatrix}$

To calculate $\det(\mathbf{A}_1)$:

$$\begin{array}{ccccc} 3 & 1 & 1 & 3 & 1 \\ -2 & 1 & -1 & -2 & 1 \\ 6 & -2 & 1 & 6 & -2 \end{array}$$

$\det(\mathbf{A}_1) = (3 - 6 + 4) - (6 + 6 - 2) = 1 - 10 = -9$

$\mathbf{A}_2 = \begin{pmatrix} 2 & 3 & 1 \\ 4 & -2 & -1 \\ 2 & 6 & 1 \end{pmatrix}$

To calculate $\det(\mathbf{A}_2)$:

$$\begin{array}{ccccc} 2 & 3 & 1 & 2 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 2 & 6 & 1 & 2 & 6 \end{array}$$

$\det(\mathbf{A}_2) = (-4 - 6 + 24) - (-4 - 12 + 12) = 14 + 4 = 18$

$\mathbf{A}_3 = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \\ 2 & -2 & 6 \end{pmatrix}$

To calculate $\det(\mathbf{A}_3)$:

$$\begin{array}{ccccc} 2 & 1 & 3 & 2 & 1 \\ 4 & 1 & -2 & 4 & 1 \\ 2 & -2 & 6 & 2 & -2 \end{array}$$

$\det(\mathbf{A}_3) = (12 - 4 - 24) - (6 + 8 + 24) = 1 - 10 = -16 - 38 = -54$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-9}{-18} = \frac{1}{2}$$
$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{18}{-18} = -1$$
$$z = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})} = \frac{-54}{-18} = 3$$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 3 \end{pmatrix}$

2. Gauss elimination method

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \qquad (*)$$

$$\mathbf{A}\ \mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss elimination method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right)$$

2. Use **elementary row operations** on the augmented matrix to transform the matrix $\mathbf{A}$ to an upper triangular matrix with leading coeficient of each row equals 1.

   (Note: the leading coefficient of a row is the leftmost non-zero element of that row).

$$\left(\begin{array}{cccccc|c} 1 & c_{12} & c_{13} & c_{14} & \dots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & c_{24} & \dots & a_{2n} & d_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & c_{(n-1)n} & d_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 1 & d_n \end{array}\right)$$

3. From the last augmented matrix , $x_n = d_n$ and the rest of the unknowns can be calculated by backward substitution.

Example 1: Use Gauss elimination method to solve the system

$$\begin{array}{ccccccc} x & - & 2y & + & z & = & 4 \\ -x & + & 2y & + & z & = & -2 \\ 4x & - & 3y & - & z & = & -4 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 4 & -3 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array}\right) \xrightarrow{-4R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, $z = 1$.
$y - z = -4 \;\Rightarrow\; y - 1 = -4 \;\Rightarrow\; y = -4 + 1 = -3$
$x - 2y + z = 4 \;\Rightarrow\; x - 2(-3) + 1 = 4 \;\Rightarrow\; x + 6 + 1 = 4 \;\Rightarrow\; x = 4 - 7 = -3$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

Example 2: Use Gauss elimination method to solve the system

$$\begin{array}{ccccccccc} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right) \xrightarrow{-3R_1+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefor, $w = 1$

$z - \dfrac{1}{3}w = -\dfrac{1}{3} \ \Rightarrow \ z - \dfrac{1}{3} = -\dfrac{1}{3} \ \Rightarrow \ z = 0$

$y + \dfrac{1}{2}z = -1 \ \Rightarrow \ y + \dfrac{1}{2}(0) = -1 \ \Rightarrow \ y = -1$

$x + y + z - w = 0 \ \Rightarrow \ x - 1 + 0 - 1 = 0 \ \Rightarrow \ x = 2$

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

3. Gauss-Jordan method

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \qquad (*)$$

$$\mathbf{A}\ \mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss-Jordan method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right)$$

2. Use elementary row operations on the augmented matrix to transform the matrix $\mathbf{A}$ to the identity matrix .

$$\left(\begin{array}{cccccc|c} 1 & 0 & \dots & 0 & 0 & & d_1 \\ 0 & 1 & \dots & 0 & 0 & & d_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & & d_{n-1} \\ 0 & 0 & \dots & 0 & 1 & & d_n \end{array}\right)$$

3. From the last augmented matrix , $x_i = d_i$ for every $i = 1, 2, \cdots, n$

Example 1: Use Gauss-Jordan method to solve the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & - & y & + & 2z & = & 0 \\ 2x & & & + & z & = & 2 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_3+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Therefore , $x = 1$, $y = 1$ and $z = 0$.

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Example 2: Use Gauss-Jordan method to solve the system

$$\begin{array}{ccccccccc} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : (Note : This is example 2 in Gauss elimination method)
The augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right) \xrightarrow{-3R_1+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_4+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_4+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_3+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, $x = 2$, $y = -1$, $z = 0$ and $w = 1$.

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

CHAPTER FOUR

INTEGRATION

1. Indefinite integral

2. Integration by substitution

3. Integration by parts

4. Integration of rational functions
(Method of partial fractions)

1. Indefinite integral

Definition (Antiderivative): A function G is called an antiderivative of the function f on the interval $[a, b]$ if $G'(x) = f(x)$ for all $x \in [a, b]$.

Examples : What is the antiderivative of the following functions

1. $f(x) = 2x$.
2. $f(x) = \cos x$.
3. $f(x) = \sec^2 x$
4. $f(x) = \dfrac{1}{x}$
5. $f(x) = e^x$

Solution :

1. $G(x) = x^2 + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\left(x^2 + c\right) = 2x + 0 = 2x$$
2. $G(x) = \sin x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\sin x + c) = \cos x$$
3. $G(x) = \tan x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\tan x + c) = \sec^2 x$$
4. $G(x) = \ln|x| + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\ln|x| + c) = \frac{1}{x}$$
5. $G(x) = e^x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(e^x + c) = e^x$$

Note: If $G_1(x)$ and $G_2(x)$ are both antiderivatives of the function $f(x)$ then $G_1(x) - G_2(x) = \text{constant}$.

Definition (indefinite integral): If $G(x)$ is the antiderivative of $f(x)$ then $\displaystyle\int f(x)\ dx = G(x) + c$, $\displaystyle\int f(x)\ dx$ is called the indefinite integral of the function $f(x)$.

Basic Rules of integration :

1. $\displaystyle\int 1 \ dx = x + c$
2. $\displaystyle\int x^n \ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$
3. $\displaystyle\int \cos x \ dx = \sin x + c$
4. $\displaystyle\int \sin x \ dx = -\cos x + c$
5. $\displaystyle\int \sec^2 x \ dx = \tan x + c$
6. $\displaystyle\int \csc^2 x \ dx = -\cot x + c$
7. $\displaystyle\int \sec x \ \tan x \ dx = \sec x + c$
8. $\displaystyle\int \csc x \ \cot x = -\csc x + c$
9. $\displaystyle\int \frac{1}{x} \ dx = \ln|x| + c$
10. $\displaystyle\int e^x \ dx = e^x + c$
11. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \ dx = \sin^{-1} x + c$, where $|x| < 1$
12. $\displaystyle\int \frac{1}{1+x^2} \ dx = \tan^{-1} x + c$
13. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} \ dx = \sec^{-1} x + c$, where $|x| > 1$

Properties of indefinite integral :

1. $\displaystyle\int k \ f(x) \ dx = k \ \int f(x) \ dx$, where $k \in \mathbb{R}$
2. $\displaystyle\int (f(x) \pm g(x)) \ dx = \int f(x) \ dx \pm \int g(x) \ dx$

Examples : Evaluate the following integrals

1. $\displaystyle\int \left(4x^2 - \frac{5}{x^3}\right)\ dx$

   Solution : $\displaystyle\int \left(4x^2 - \frac{5}{x^3}\right)\ dx = \int 4x^2\ dx - \int \frac{5}{x^3}\ dx$

   $\displaystyle= 4\int x^2\ dx - 5\int x^{-3}dx = 4\ \frac{x^3}{3} - 5\ \frac{x^{-2}}{-2} + c = \frac{4}{3}x^3 + \frac{5}{2x^2} + c$

2. $\displaystyle\int \left(3x^{\frac{1}{3}} + \frac{1}{\sqrt{x}}\right)\ dx$

   Solution : $\displaystyle\int \left(3x^{\frac{1}{3}} + \frac{1}{\sqrt{x}}\right)\ dx = 3\int x^{\frac{1}{3}}\ dx + \int x^{-\frac{1}{2}}\ dx$

   $\displaystyle= 3\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) + \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = \frac{9}{4}x^{\frac{4}{3}} + 2x^{\frac{1}{2}} + c$

3. $\displaystyle\int (2\cos x - 3\sec^2 x)\ dx$

   Solution : $\displaystyle\int (2\cos x - 3\sec^2 x)\ dx = 2\int \cos x\ dx - 3\int \sec^2 x\ dx$

   $= 2\sin x - 3\tan x + c$

4. $\displaystyle\int (7\sec x\ \tan x + 5\csc^2 x)\ dx$

   Solution : $\displaystyle\int (7\sec x\ \tan x + 5\csc^2 x)\ dx = 7\int \sec x \tan x\ dx + 5\int \csc^2 x\ dx$

   $= 7\sec x + 5(-\cot x) + c = 7\sec x - 5\cot x + c$

5. $\displaystyle\int \left(\frac{2}{x} - \frac{3}{x^2}\right)\ dx$

   Solution : $\displaystyle\int \left(\frac{2}{x} - \frac{3}{x^2}\right)\ dx = 2\int \frac{1}{x}\ dx - 3\int x^{-2}\ dx$

   $\displaystyle= 2\ln|x| - 3\left(\frac{x^{-1}}{-1}\right) + c = 2\ln|x| + \frac{3}{x} + c$

6. $\displaystyle\int \left(9e^x - \frac{3}{1+x^2}\right)\, dx$

Solution : $\displaystyle\int \left(9e^x - \frac{3}{1+x^2}\right)\, dx = 9\int e^x\, dx - 3\int \frac{1}{1+x^2}\, dx$

$= 9e^x - 3\tan^{-1} x + c$

7. $\displaystyle\int \left(\frac{4}{\sqrt{1-x^2}} + \frac{1}{\sqrt[3]{x}}\right)\, dx$

Solution : $\displaystyle\int \left(\frac{4}{\sqrt{1-x^2}} + \frac{1}{\sqrt[3]{x}}\right)\, dx = 4\int \frac{1}{\sqrt{1-x^2}}\, dx + \int x^{-\frac{1}{3}}\, dx$

$= 4\sin^{-1} x + \left(\dfrac{x^{\frac{2}{3}}}{\frac{2}{3}}\right) + c = 4\sin^{-1} x + \dfrac{3}{2}x^{\frac{2}{3}} + c$

The definite integral :
If f is a continuous function on the interval $[a, b]$ and G is the antiderivative of f on $[a, b]$ then the definite integral of f on $[a, b]$ is

$$\int_a^b f(x)\, dx = [G(x)]_a^b = G(b) - G(a)$$

Examples : Evaluate the following integrals :

1. $\displaystyle\int_1^3 (3x^2 + 5)dx$

Solution : $\displaystyle\int_1^3 (3x^2 + 5)dx = \left[x^3 + 5x\right]_1^3$

$= \left(3^3 + 5 \times 3\right) - \left(1^3 + 5 \times 1\right) = (27 + 15) - (1 + 5) = 36$

2. $\displaystyle\int_0^1 (2x + e^x)\, dx$

Solution : $\displaystyle\int_0^1 (2x + e^x)\, dx = \left[x^2 + e^x\right]_0^1$

$= \left(1^2 + e^1\right) - \left(0^2 + e^0\right) = 1 + e - 1 = e$

2. Integration by substitution

The main idea of integration by substitution is to use a suitable substitution to transform the given integral to an easier integral that can be solved by one of the basic rules of integration.

Example : Evaluate the integral $\int x(x^2+3)^6\ dx$

Solution : Use the substitution $u = x^2+3$

Then $du = 2x\ dx \ \Rightarrow\ \frac{1}{2}\ du = x\ dx$

$\int x(x^2+3)^6\ dx = \int u^6 \frac{1}{2}\ du = \frac{1}{2}\int u^6\ du$

$= \frac{1}{2}\ \frac{u^7}{7} + c = \frac{(x^2+3)^7}{14} + c$

By the chain rule $\frac{d}{dx}\,[f(x)]^{n+1} = (n+1)\ \ [f(x)]^n\ \ f'(x)$, where $n \neq -1$

Hence $\int [f(x)]^n\ \ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

So, the above integral can be solved as follows

$\int x(x^2+3)^6\ dx = \frac{1}{2}\int (x^2+3)^6\ (2x)\ dx = \frac{1}{2}\ \frac{(x^2+3)^7}{7} + c$

Basic rules of integrations and their general forms :

1. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$

   $\int [f(x)]^n\, f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

2. $\int \frac{1}{x}\ dx = \ln|x| + c$

   $\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + c$

3. $\int e^x\ dx = e^x + c$

   $\int e^{f(x)} f'(x)\ dx = e^{f(x)} + c$

4. $\int \cos x\ dx = \sin x + c$

   $\int \cos\left(f(x)\right) f'(x)\ dx = \sin\left(f(x)\right) + c$

5. $\displaystyle\int \sin x \; dx = -\cos x + c$

$\displaystyle\int \sin(f(x))\, f'(x) \; dx = -\cos(f(x)) + c$

6. $\displaystyle\int \sec^2 x \; dx = \tan x + c$

$\displaystyle\int \sec^2(f(x))\, f'(x) \; dx = \tan(f(x)) + c$

7. $\displaystyle\int \csc^2 x \; dx = -\cot x + c$

$\displaystyle\int \csc^2(f(x))\, f'(x) \; dx = -\cot(f(x)) + c$

8. $\displaystyle\int \sec x \tan x \; dx = \sec x + c$

$\displaystyle\int \sec(f(x)) \tan(f(x))\, f'(x) \; dx = \sec(f(x)) + c$

9. $\displaystyle\int \csc x \cot x \; dx = -\csc x + c$

$\displaystyle\int \csc(f(x)) \cot(f(x))\, f'(x) \; dx = -\csc(f(x)) + c$

10. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, where $a > 0$ and $|x| < a$

$\displaystyle\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} \; dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$ and $|f(x)| < a$

11. $\displaystyle\int \frac{1}{a^2 + x^2} \; dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$, where $a > 0$

$\displaystyle\int \frac{f'(x)}{a^2 + [f(x)]^2} \; dx = \frac{1}{a} \tan^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

12. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}} \; dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$, where $a > 0$ and $|x| > a$

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} \; dx = \frac{1}{a} \sec^{-1}\left(\frac{f(x)}{a}\right) + c$, where $|f(x)| > a$

Examples : Evaluate the following integrals

1. $\displaystyle\int (x^2+2x)(x^3+3x^2+5)^{10}\ dx$

   Solution :

   $$\int (x^2+2x)(x^3+3x^2+5)^{10}\ dx = \frac{1}{3}\int (x^3+3x^2+5)^{10}\left[3(x^2+2x)\right]\ dx$$

   $$= \frac{1}{3}\int (x^3+3x^2+5)^{10}(3x^2+6x)\ dx = \frac{1}{3}\ \frac{(x^3+3x^2+5)^{11}}{11} + c$$

2. $\displaystyle\int \frac{x+1}{(x^2+2x+6)^5}\ dx$

   Solution :

   $$\int \frac{x+1}{(x^2+2x+6)^5}\ dx = \int (x^2+2x+6)^{-5}(x+1)\ dx$$

   $$= \frac{1}{2}\int (x^2+2x+6)^{-5}(2x+2)\ dx = \frac{1}{2}\ \frac{(x^2+2x+6)^{-4}}{-4} + c$$

3. $\displaystyle\int \frac{x^3+x}{\sqrt{x^4+2x^2+5}}\ dx$

   Solution :

   $$\int \frac{x^3+x}{\sqrt{x^4+2x^2+5}}\ dx = \int (x^4+2x^2+5)^{-\frac{1}{2}}(x^3+x)\ dx$$

   $$= \frac{1}{4}\int (x^4+2x^2+5)^{-\frac{1}{2}}(4x^3+4x)\ dx = \frac{1}{4}\ \frac{(x^4+2x^2+5)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

4. $\displaystyle\int \frac{x^2+1}{x^3+3x+8}\ dx$

   Solution :

   $$\int \frac{x^2+1}{x^3+3x+8}\ dx = \frac{1}{3}\int \frac{3(x^2+1)}{x^3+3x+8}\ dx$$

   $$= \frac{1}{3}\int \frac{3x^2+3}{x^3+3x+8}\ dx = \frac{1}{3}\ln|x^3+3x+8| + c$$

5. $\displaystyle\int \frac{\sin x}{1+\cos x}\ dx$

   Solution :

   $$\int \frac{\sin x}{1+\cos x}\ dx = -\int \frac{-\sin x}{1+\cos x}\ dx = -\ln|1+\cos x| + c$$

6. $\displaystyle\int \frac{e^{5x}}{e^{5x}-2}\ dx$

Solution :

$$\int \frac{e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\int \frac{5e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\ln|e^{5x}-2| + c$$

7. $\displaystyle\int (3x^2+1)\sin(x^3+x+1)\ dx$

Solution :

$$\int (3x^2+1)\sin(x^3+x+1)\ dx = \int \sin(x^3+x+1)\ (3x^2+1)\ dx$$

$$= -\cos(x^3+x+1) + c$$

8. $\displaystyle\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx$

Solution :

$$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx = \int \sec^2\sqrt{x}\ \frac{1}{\sqrt{x}}\ dx$$

$$= 2\int \sec^2\sqrt{x}\ \frac{1}{2\sqrt{x}}\ dx = 2\tan\sqrt{x} + c$$

9. $\displaystyle\int x\ \csc(x^2+2)\cot(x^2+2)\ dx$

Solution :

$$\int x\ \csc(x^2+2)\cot(x^2+2)\ dx = \int \csc(x^2+2)\cot(x^2+2)\ x\ dx$$

$$\frac{1}{2}\int \csc(x^2+2)\cot(x^2+2)\ (2x)\ dx = -\frac{1}{2}\csc(x^2+2) + c$$

10. $\displaystyle\int e^{7\sin x}\cos x\ dx$

Solution :

$$\int e^{7\sin x}\cos x\ dx = \frac{1}{7}\int e^{7\sin x}(7\cos x)\ dx = \frac{1}{7}e^{7\sin x} + c$$

11. $\displaystyle\int \frac{e^{\frac{3}{x}}}{x^2}\ dx$

Solution :

$$\int \frac{e^{\frac{3}{x}}}{x^2}\ dx = \int e^{\frac{3}{x}}\ \frac{1}{x^2}\ dx$$

$$= -\frac{1}{3}\int e^{\frac{3}{x}}\ \frac{-3}{x^2}\ dx = -\frac{1}{3}e^{\frac{3}{x}} + c$$

12. $\displaystyle\int \frac{x}{\sqrt{9-x^4}}\ dx$

Solution :

$$\int \frac{x}{\sqrt{9-x^4}}\ dx = \int \frac{x}{\sqrt{3^2-(x^2)^2}}\ dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}\ dx = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c$$

13. $\displaystyle\int \frac{1}{x^2-6x+10}\ dx$

Solution

$$\int \frac{1}{x^2-6x+10}\ dx = \int \frac{1}{(x^2-6x+9)+(10-9)}\ dx$$

$$= \int \frac{1}{(x-3)^2+1}\ dx = \tan^{-1}(x-3) + c$$

14. $\displaystyle\int \frac{3}{x^2+2x+5}\ dx$

Solution

$$\int \frac{3}{x^2+2x+5}\ dx = \int \frac{3}{(x^2+2x+1)+(5-1)}\ dx$$

$$= 3\int \frac{1}{(x+1)^2+2^2}\ dx = 3\ \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

15. $\displaystyle\int \frac{1}{x\ln|x|}\ dx$

Solution

$$\int \frac{1}{x\ln|x|}\ dx = \int \frac{\frac{1}{x}}{\ln|x|}\ dx = \ln|\ln|x|| + c$$

16. $\displaystyle\int \frac{2x-1}{x^2+1}\ dx$

Solution :

$$\int \frac{2x-1}{x^2+1}\ dx = \int \frac{2x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx$$

$$= \ln(x^2+1) - \tan^{-1}x + c$$

3. Integration by parts

It is used to solve an integral of a product of two functions using the formula

$$\int u\ dv = u\ v - \int v\ du$$

Examples : Evaluate the following integrals

1. $\displaystyle\int xe^x\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = x & dv = e^x\ dx \\ du = dx & v = e^x \end{array}$$

   $$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + c$$

2. $\displaystyle\int x^2 \sin x\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = x^2 & dv = \sin x\ dx \\ du = 2xdx & v = -\cos x \end{array}$$

   $$\int x^2 \sin x\ dx = -x^2 \cos x - \int 2x(-\cos x)\ dx$$

   $$= -x^2 \cos x + 2 \int x \cos x\ dx$$

   Using integration by parts again

   $$\begin{array}{ll} u = x & dv = \cos x\ dx \\ du = dx & v = \sin x \end{array}$$

   $$\int x^2 \sin x\ dx = -x^2 \cos x + 2 \left(x \sin x - \int \sin x\ dx \right)$$

   $$= -x^2 \cos x + 2\left(x \sin x - (-\cos x)\right) + c$$

   $$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

3. $\displaystyle\int x \ln|x|\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = \ln|x| & dv = x\ dx \\ du = \dfrac{1}{x}\ dx & v = \dfrac{x^2}{2} \end{array}$$

$$\int x \ln|x| \; dx = \frac{x^2}{2} \ln|x| - \int \frac{1}{x} \; \frac{x^2}{2} \; dx$$

$$= \frac{x^2}{2} \ln|x| - \frac{1}{2} \int x \; dx = \frac{x^2}{2} \ln|x| - \frac{1}{2} \; \frac{x^2}{2} + c = \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + c$$

4. $\displaystyle\int \ln|x| \; dx$

Solution : Using integration by parts

$$\begin{array}{ll} u = \ln|x| & dv = dx \\ du = \dfrac{1}{x} \; dx & v = x \end{array}$$

$$\int \ln|x| \; dx = x \ln|x| - \int x \; \frac{1}{x} \; dx = x \ln|x| - \int 1 \; dx$$

$$= x \ln|x| - x + c$$

5. $\displaystyle\int \tan^{-1} x \; dx$

Solution : Using integration by parts

$$\begin{array}{ll} u = \tan^{-1} x & dv = dx \\ du = \dfrac{1}{1+x^2} \; dx & v = x \end{array}$$

$$\int \tan^{-1} x \; dx = x \tan^{-1} x - \int x \; \frac{1}{1+x^2} \; dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \; dx = x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + c$$

6. $\displaystyle\int \sin^{-1} x \; dx$

Solution : Using integration by parts

$$\begin{array}{ll} u = \sin^{-1} x & dv = dx \\ du = \dfrac{1}{\sqrt{1-x^2}} \; dx & v = x \end{array}$$

$$\int \sin^{-1} x \; dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \; dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) \; dx = x \sin^{-1} x + \frac{1}{2} \; \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

7. $\int e^x \sin x \; dx$

Solution : Using integration by parts

$$\begin{array}{ll} u = \sin x & dv = e^x \; dx \\ du = \cos x \; dx & v = e^x \end{array}$$

$$\int e^x \sin x \; dx = e^x \sin x - \int e^x \cos x \; dx$$

Using integration by parts again

$$\begin{array}{ll} u = \cos x & dv = e^x \; dx \\ du = -\sin x \; dx & v = e^x \end{array}$$

$$\int e^x \sin x \; dx = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \; dx \right)$$

$$\int e^x \sin x \; dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \; dx \right)$$

$$\int e^x \sin x \; dx = e^x \sin x - e^x \cos x - \int e^x \sin x \; dx$$

$$2 \int e^x \sin x \; dx = e^x \sin x - e^x \cos x + c$$

$$\int e^x \sin x \; dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x + c \right)$$

4. Integral of rational functions
(The metod of partial fractions)

Method of partial fractions is used to solve integrals of the form $\int \frac{P(x)}{Q(x)}\, dx$ where $P(x)$, $Q(x)$ are polynomials and degree $P(x) <$ degree $Q(x)$.
If degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials .

Definition (linear factor) :
A linear factor is a polynomial of degree 1.
It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

Examples :
x , $3x$, $2x - 7$ are examples of linear factors .

Definition (irreducible quadratic) :
An irreducible quadratic is a polynomial of degree 2.
It has the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2 - 4ac < 0$.

Examples :

1. $x^2 + 9$ and $x^2 + x + 1$ are examples of irreducible quadratics.
2. $x^2 = x\ x$ and $x^2 - 1 = (x - 1)(x + 1)$ are reducible quadratics .

How to write $\frac{P(x)}{Q(x)}$ as partial fractions decomposition ?
Write $Q(x)$ as a product of linear factors and irreducible quadratics (if possible).
If $Q(x) = (a_1x + a_2)^m\ (b_1x^2 + b_2x + b_3)^n$ where $m, n \in \mathbb{N}$ then

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \cdots + \frac{A_m}{(a_1x + a_2)^m} \\ &+ \frac{B_1x + C_1}{b_1x^2 + b_2x + b_3} + \frac{B_2x + C_2}{(b_1x^2 + b_2x + b_3)^2} + \cdots + \frac{B_nx + C_n}{(b_1x^2 + b_2x + b_3)^n}\end{aligned}$$

Where $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n, C_1, C_2, \cdots, C_n \in \mathbb{R}$.

Examples : Write the partial fractions decomposition of the follwoing

1. $\frac{2x + 6}{x^2 - 2x - 3}$

   Solution :

   $$\frac{2x + 6}{x^2 - 2x - 3} = \frac{2x + 6}{(x - 3)(x + 1)} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}$$

2. $\dfrac{x+5}{x^2+4x+4}$

Solution :

$$\frac{x+5}{x^2+4x+4} = \frac{x+5}{(x+2)^2} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2}$$

3. $\dfrac{x^2+1}{x^4+4x^2}$

Solution :

$$\frac{x^2+1}{x^4+4x^2} = \frac{x^2+1}{x^2(x^2+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x+C_1}{x^2+4}$$

4. $\dfrac{2x+7}{(x+1)(x^2+9)^2}$

Solution :

$$\frac{2x+7}{(x+1)(x^2+9)^2} = \frac{A_1}{x+1} + \frac{B_1x+C_1}{x^2+9} + \frac{B_2x+C_2}{(x^2+9)^2}$$

5. $\dfrac{x}{(x-1)(x^2-1)}$

Solution :

$$\frac{x}{(x-1)(x^2-1)} = \frac{x}{(x+1)(x-1)^2} = \frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}$$

6. $\dfrac{x^3+x}{x^2-1}$

Solution : Using long division of polynomials

$$\frac{x^3+x}{x^2-1} = \frac{(x^3-x)+2x}{x^2-1} = \frac{x(x^2-1)+2x}{x^2-1} = x + \frac{2x}{x^2-1}$$

$$\frac{x^3+x}{x^2-1} = x + \frac{2x}{(x-1)(x+1)} = x + \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

Examples : Evaluate the following integrals

1. $\displaystyle\int \frac{x+3}{(x-3)(x-2)}\,dx$

Solution : Using the method of partial fractions

$$\frac{x+3}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

$$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2)}{(x-3)(x-2)} + \frac{A_2(x-3)}{(x-2)(x-3)}$$

$$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2)+A_2(x-3)}{(x-3)(x-2)}$$

$x+3 = A_1(x-2)+A_2(x-3) = A_1x-2A_1+A_2x-3A_2$

$x+3 = (A_1+A_2)\,x+(-2A_1-3A_2)$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 & + & A_2 & = & 1 & \longrightarrow (1) \\ -2A_1 & - & 3A_2 & = & 3 & \longrightarrow (2) \end{cases}$$

Muliplying equation (1) by 2 and adding it to equation (2) :

$-A_2 = 5 \implies A_2 = -5$

From Equation (1) : $A_1 - 5 = 1 \implies A_1 = 1+5 = 6$

$$\frac{x+3}{(x-3)(x-2)} = \frac{6}{x-3} + \frac{-5}{x-2}$$

$$\int \frac{x+3}{(x-3)(x-2)}\, dx = \int \left(\frac{6}{x-3} - \frac{5}{x-2}\right)\, dx$$

$$= 6\int \frac{1}{x-3}\, dx - 5\int \frac{1}{x-2}\, dx = 6\ln|x-3| - 5\ln|x-2| + c$$

2. $\displaystyle\int \frac{x+1}{x^2-1}\, dx$

Solution :

$$\int \frac{x+1}{x^2-1}\, dx = \int \frac{x+1}{(x-1)(x+1)}\, dx$$

$$= \int \frac{1}{x-1}\, dx = \ln|x-1| + c$$

3. $\displaystyle\int \frac{x-1}{(x+1)(x+2)^2}\, dx$

Solution : Using the method of partial fractions

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}$$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1(x+2)^2}{(x+1)(x+2)^2} + \frac{A_2(x+1)(x+2)}{(x+1)(x+2)^2} + \frac{A_3(x+1)}{(x+1)(x+2)^2}$$

$x-1 = A_1(x+2)^2 + A_2(x+1)(x+2) + A_3(x+1)$

$x-1 = A_1(x^2+4x+4) + A_2(x^2+3x+2) + A_3(x+1)$

$x-1 = A_1x^2 + 4A_1x + 4A_1 + A_2x^2 + 3A_2x + 2A_2 + A_3x + A_3$

$x-1 = (A_1+A_2)\,x^2 + (4A_1+3A_2+A_3)\,x + (4A_1+2A_2+A_3)$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 & + & A_2 & & & = & 0 & \longrightarrow (1) \\ 4A_1 & + & 3A_2 & + & A_3 & = & 1 & \longrightarrow (2) \\ 4A_1 & + & 2A_2 & + & A_3 & = & -1 & \longrightarrow (3) \end{cases}$$

Subtracting equation (3) from equation (2) : $A_2 = 2$

From equation (1) : $A_1 + 2 = 0 \;\Rightarrow\; A_1 = -2$

From equation (2) :

$(4 \times -2) + (3 \times 2) + A_3 = 1 \;\Rightarrow\; -8 + 6 + A_3 = 1 \;\Rightarrow\; A_3 = 3$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

$$\int \frac{x-1}{(x+1)(x+2)^2}\, dx = \int \left(\frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2} \right) dx$$

$$= -2 \int \frac{1}{x+1}\, dx + 2 \int \frac{1}{x+2}\, dx + 3 \int (x+2)^{-2}\, dx$$

$$= -2\ln|x+1| + 2\ln|x+2| + 3\, \frac{(x+2)^{-1}}{-1} + c$$

$$= -2\ln|x+1| + 2\ln|x+2| - \frac{3}{x+2} + c$$

4. $\displaystyle\int \frac{2x^2+3x+2}{x^3+x}\, dx$

Solution : Using the method of partial functions

$$\frac{2x^2+3x+2}{x^3+x} = \frac{2x^2+3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x^2+3x+2}{x^3+x} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{x(Bx+C)}{x(x^2+1)}$$

$$2x^2+3x+2 = A(x^2+1) + x(Bx+C) = Ax^2 + A + Bx^2 + Cx$$

$$2x^2+3x+2 = (A+B)x^2 + Cx + A$$

By comparing the coefficients of the polynomials

$$\begin{cases} A+B & = & 2 & \longrightarrow (1) \\ C & = & 3 & \longrightarrow (2) \\ A & = & 2 & \longrightarrow (3) \end{cases}$$

From equation (1) : $2 + B = 2 \;\Rightarrow\; B = 0$

$$\frac{2x^2+3x+2}{x^3+x} = \frac{2}{x} + \frac{3}{x^2+1}$$

$$\int \frac{2x^2+3x+2}{x^3+x}\ dx = \int \left(\frac{2}{x} + \frac{3}{x^2+1}\right)\ dx$$

$$= 2\int \frac{1}{x}\ dx + 3\int \frac{1}{x^2+1}\ dx$$

$$= 2\ln|x| + 3\tan^{-1} x + c$$

CHAPTER FIVE

APPLICATIONS OF INTEGRATION

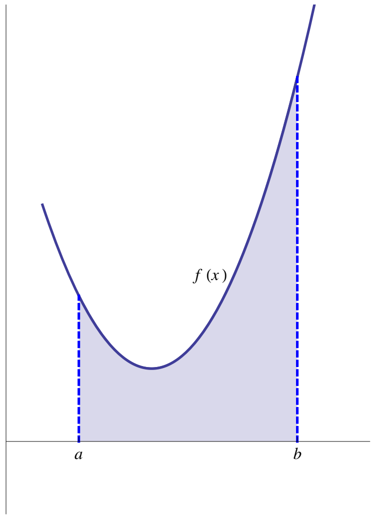
1. Area

2. Volume of a solid of revolution (using disk or washer method)

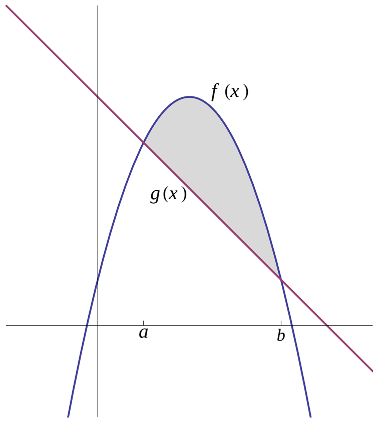
3. Volume of a solid of revolution (using cylindrical shells method)

4. Polar Coordinates and Applications

1. Area



In the above figure the area under the graph of $f(x)$ on the interval $[a, b]$ is given by the definite integral $\int_a^b f(x)\ dx$

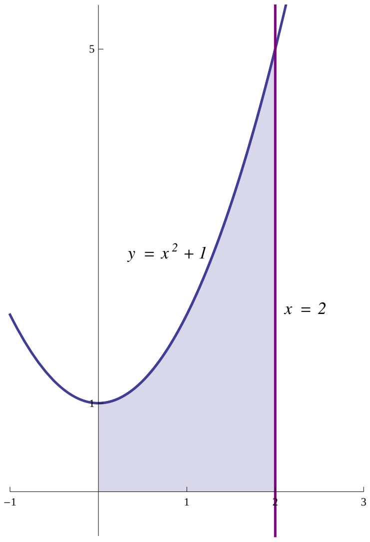


In the above figure the graphs of $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$.
The area bounded by the graphs of the curves of $f(x)$ and $g(x)$ equals

$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b [f(x) - g(x)]\ dx$$

Examples :

1. Find the area of the region bounded by the graphs of $x = 0$, $y = 0$, $x = 2$ and $y = x^2 + 1$



$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

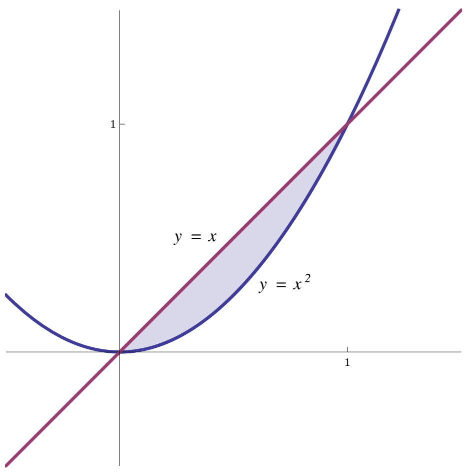
$x = 0$ is the y-axis and $y = 0$ is the x-axis.

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$

$$\text{Area} = \int_0^2 (x^2 + 1)\, dx = \left[\frac{x^3}{3} + x\right]_0^2$$

$$\text{Area} = \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right) = \frac{8}{3} + 2 = \frac{14}{3}$$

2. Find the area of the region bounded by the graphs of $y = x$ and $y = x^2$



$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = x$ is a straight line passing through the origin with slope equals 1.

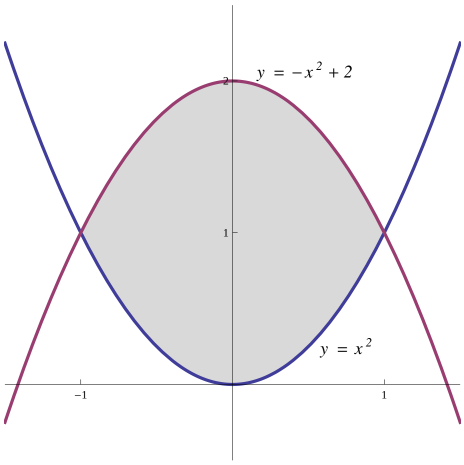
Points of intersection of $y = x^2$ and $y = x$:

$x^2 = x \;\Rightarrow\; x^2 - x = 0 \;\Rightarrow\; x(x-1) = 0 \;\Rightarrow\; x = 0\;,\; x = 1$

$$\text{Area} = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$\text{Area} = \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. Find the area of the region bounded by the graphs of $y = x^2$ and $y = -x^2 + 2$



$y = -x^2 + 2$ is a parabola with vertex $(0, 2)$ and opens downwards

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

Points of intersection of $y = x^2$ and $y = -x^2 + 2$:

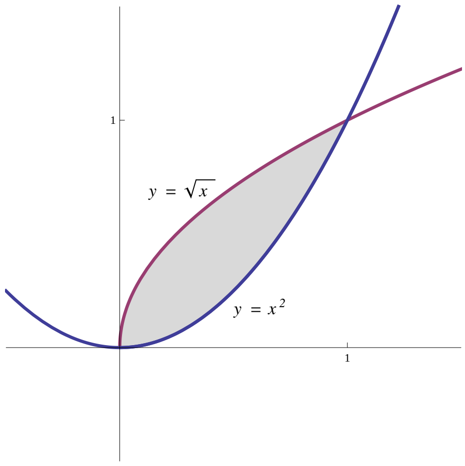
$x^2 = -x^2 + 2 \;\Rightarrow\; 2x^2 = 2 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1$

$$\text{Area} = \int_{-1}^1 \left[(-x^2 + 2) - x^2\right]\, dx = \int_{-1}^1 (2 - 2x^2)\, dx$$

$$\text{Area} = \left[2x - \frac{2x^3}{3}\right]_{-1}^1 = \left[\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right]$$

$$\text{Area} = 2 - \frac{2}{3} + 2 - \frac{2}{3} = 4 - \frac{4}{3} = \frac{12 - 4}{3} = \frac{8}{3}$$

4. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$



$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0, 0)$ and opens to the right.

Points of intersection of $y = x^2$ and $y = \sqrt{x}$:

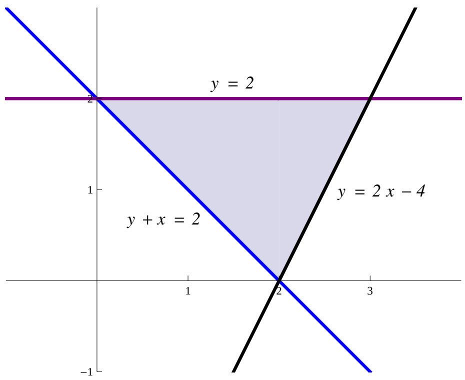
$x^2 = \sqrt{x} \;\Rightarrow\; x^4 = x \;\Rightarrow\; x^4 - x = 0 \;\Rightarrow\; x(x^3 - 1) = 0$

$\Rightarrow\; x = 0 \;, x^3 = 1 \;\Rightarrow\; x = 0 \;,\; x = 1$

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$\text{Area} = \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{3}$$

5. Find the area of the region bounded by the graphs of $x + y = 2$, $y = 2$ and $y = 2x - 4$



$y = 2$, $y = 2x - 4$ and $y = -x + 2$ are three straight lines.

Point of intersection of $y = 2$ and $y = -x + 2$:

$-x + 2 = 2 \ \Rightarrow \ x = 0$

$y = 2$ and $y = -x + 2$ intersect at the point $(0, 2)$.

Point of intersection of $y = 2$ and $y = 2x - 4$:

$2x - 4 = 2 \ \Rightarrow \ x = 3$

$y = 2$ and $y = 2x - 4$ intersect at the point $(3, 2)$

Point of intersection of $y = -x + 2$ and $y = 2x - 4$:

$2x - 4 = -x + 2 \ \Rightarrow \ 3x = 6 \ \Rightarrow \ x = 2$

$y = -x + 2$ and $y = 2x - 4$ intersect at the point $(2, 0)$.

$$\text{Area} = \int_0^2 [2 - (-x + 2)] \ dx + \int_2^3 [2 - (2x - 4)] \ dx$$

$$\text{Area} = \int_0^2 x \ dx + \int_2^3 (6 - 2x) \ dx = \left[\frac{x^2}{2}\right]_0^2 + \left[6x - x^2\right]_2^3$$

$$\text{Area} = \left[\frac{2^2}{2} - \frac{0^2}{2}\right] + \left[(6 \times 3 - 3^2) - (6 \times 2 - 2^2)\right]$$

$$\text{Area} = (2 - 0) + [(18 - 9) - (12 - 4)] = 2 + (9 - 8) = 2 + 1 = 3$$

Another solution :

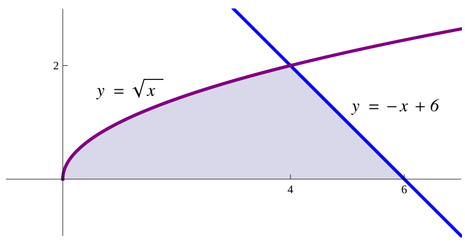
$y + x = 2 \ \Rightarrow \ x = -y + 2$ and $y = 2x - 4 \ \Rightarrow \ 2x = y + 4 \ \Rightarrow \ x = \frac{1}{2}y + 2$

$$\text{Area} = \int_0^2 \left[\left(\frac{1}{2}y + 2\right) - (-y + 2)\right] \ dy$$

$$\text{Area} = \int_0^2 \left(\frac{1}{2}y + y\right) \ dy = \int_0^2 \frac{3}{2}y \ dy$$

$$\text{Area} = \frac{3}{2}\left[\frac{y^2}{2}\right]_0^2 = \frac{3}{2}\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = \frac{3}{2} \times 2 = 3$$

6. Find the area of the region bounded by the graphs of $y = 0$, $y = -x + 6$ and $y = \sqrt{x}$



$y = -x + 6$ is a straight line passing through $(0,6)$ with slope equals -1.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0,0)$ and opens to the right.

Points of intersection of $x = y^2$ and $x = -y + 6$:

$y^2 = -y + 6 \;\Rightarrow\; y^2 + y - 6 = 0 \;\Rightarrow\; (y-2)(y+3) = 0 \;\Rightarrow\; y = 2\ ,\ y = -3$

(Note that $y = -3$ is not in the desired region).

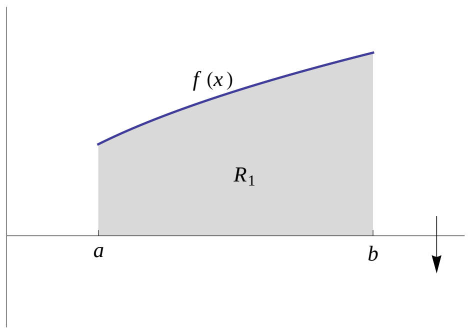
$$\text{Area} = \int_0^2 \left[(-y+6) - y^2\right]\, dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

$$\text{Area} = \left(12 - \frac{4}{2} - \frac{8}{3}\right) - (0-0-0) = 12 - 2 - \frac{8}{3} = 10 - \frac{8}{3} = \frac{30-8}{3} = \frac{22}{3}$$

2. Volume of a solid of revolution (using disk or washer method)

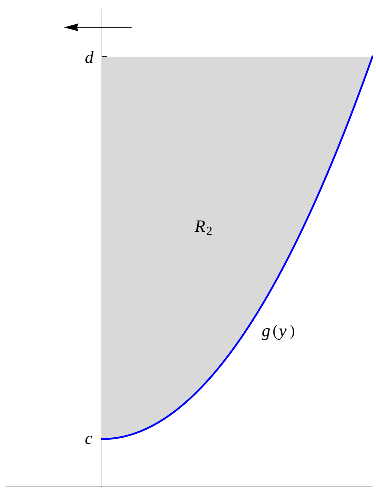
1. Disk Method

Recall that the volume of a right circular cylinder equals $\pi r^2 h$ where r is the radius of the base (which is a circle) and h is the height of the cylinder .



In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_1 around the x-axis is $V = \pi \displaystyle\int_a^b [f(x)]^2 \, dx$

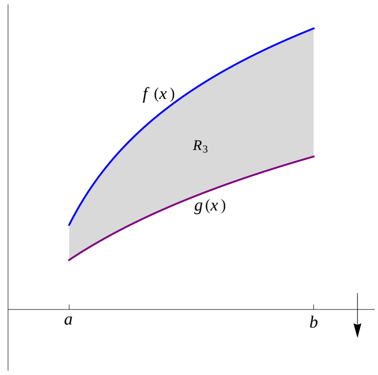


In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_2 around the y-axis is $V = \pi \displaystyle\int_c^d [g(y)]^2 \, dy$

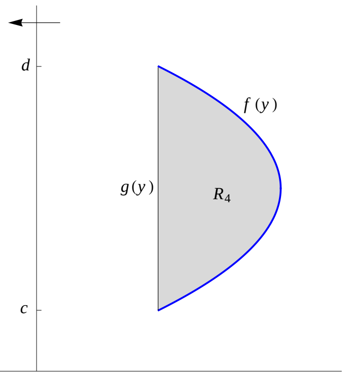
2. Washer Method

$Volume\ of\ a\ washer\ = \pi\left[(outer\ radius)^2 - (inner\ radius)^2\right]\ (thickness)$



In the above figure R_3 is the region bounded by the graphs of the curves of $f(x)$, $g(x)$, $x = a$ and $x = b$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_3 around the x-axis is $V = \pi \displaystyle\int_a^b \left[(f(x))^2 - (g(x))^2\right]\ dx$

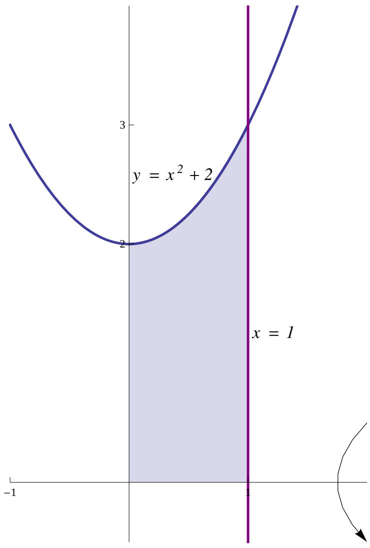


In the above figure R_4 is the region bounded by the graphs of the curves of $f(y)$ and $g(y)$, where $f(y)$ and $g(y)$ intersect at the points $y = c$ and $y = d$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_4 around the y-axis is $V = \pi \displaystyle\int_c^d \left[(f(y))^2 - (g(y))^2\right]\ dy$

Examples : Use Disk or washer method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

1. $y = x^2 + 2$, $y = 0$, $x = 0$, $x = 1$, around the x-axis



$y = x^2 + 2$ is a parabola with vertex $(0, 2)$ and opens upwards.

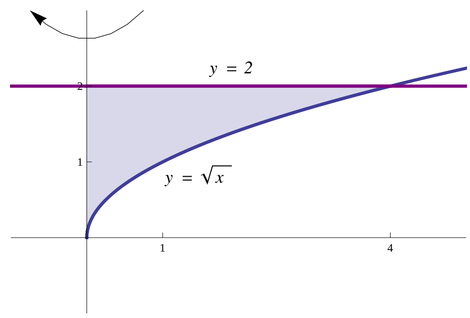
$x = 1$ is a straight line parallel to the y-axis and pasing through $(1, 0)$

Using Disk method :

$$\text{Volume} = \pi \int_0^1 (x^2+2)^2 \; dx = \pi \int_0^1 (x^4 + 4x^2 + 4) \; dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_0^1 = \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 4 \right) - (0+0+0) \right] = \frac{83\pi}{15}$$

2. $y = \sqrt{x}$, $y = 2$ and $x = 0$, around the y-axis



$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

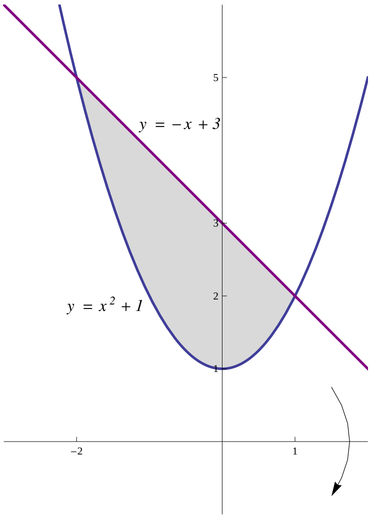
$y = 2$ is a straight line parallel to the x-axis and passing through $(0, 2)$

Using Disk method :

$$\text{Volume} = \pi \int_0^2 (y^2)^2 \ dy = \pi \int_0^2 y^4 \ dy$$

$$= \pi \left[\frac{y^5}{5} \right]_0^2 = \pi \left[\frac{2^5}{5} - 0 \right] = \frac{32\pi}{5}$$

3. $y = x^2 + 1$ and $y = -x + 3$, around the x-axis



$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

$y = -x + 3$ is a straight line with slope -1 and passing through $(0, 3)$.

Points of intersection of $y = x^2 + 1$ and $y = -x + 3$:

$x^2+1 = -x+3 \ \Rightarrow \ x^2+x-2 = 0 \ \Rightarrow \ (x+2)(x-1) = 0 \ \Rightarrow \ x = -2 \ , \ x = 1$

Using Washer method :

$$\text{volume} = \pi \int_{-2}^{1} \left[(-x+3)^2 - (x^2+1)^2 \right] \ dx$$

$$\text{Volume} = \pi \int_{-2}^{1} \left[(x^2 - 6x + 9) - (x^4 + 2x^2 + 1) \right] \ dx$$

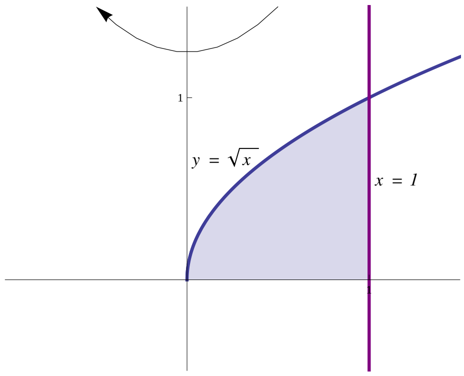
$$\text{Volume} = \pi \int_{-2}^{1} \left(-x^4 - x^2 - 6x + 8 \right) \ dx = \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right]_{-2}^{1}$$

$$= \pi \left[\left(-\frac{1}{5} - \frac{1}{3} - 3 + 8 \right) - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16 \right) \right]$$

$$= \pi \left(-\frac{1}{5} - \frac{1}{3} + 5 - \frac{32}{5} - \frac{8}{3} + 28 \right)$$

$$= \pi \left(33 - 3 - \frac{33}{5} \right) = \pi \left(30 - \frac{33}{5} \right) = \frac{150 - 33}{5} \pi = \frac{117\pi}{5}$$

4. $y = \sqrt{x}$, $y = 0$ and $x = 1$, around the y-axis



$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$

Note that $y = \sqrt{x}$ intersects $x = 1$ at the point $(1, 1)$.

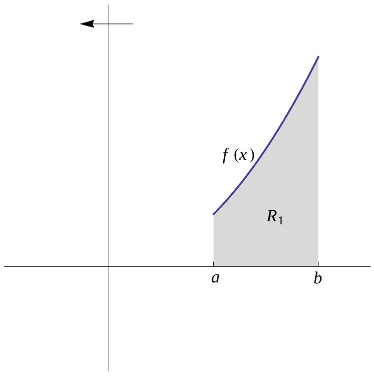
Using Washer method :

$$\text{Volume} = \pi \int_0^1 \left[(1)^2 - (y^2)^2 \right] \, dy = \pi \int_0^1 \left(1 - y^4 \right) \, dy$$

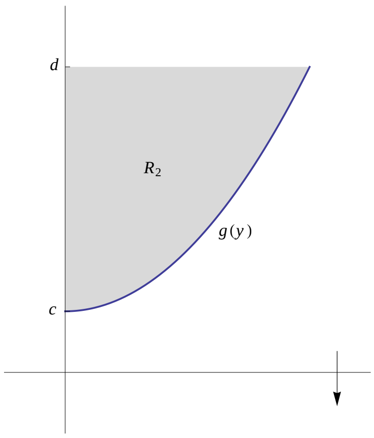
$$= \pi \left[y - \frac{y^5}{5} \right]_0^1 = \pi \left[\left(1 - \frac{1}{5} \right) - (0 - 0) \right] = \pi \left(1 - \frac{1}{5} \right) = \frac{4\pi}{5}$$

3. Volume of a solid of revolution (using cylindrical shells method)

Volume of a shell $=$ 2π (average radius) (altitude) (thickness)



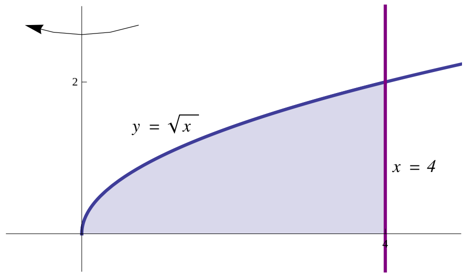
In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_1 around the y-axis is $V = 2\pi \int_a^b x\ f(x)\ dx$



In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_2 around the x-axis is $V = 2\pi \int_c^d y\ g(y)\ dy$

Examples : Use cylindrical shells method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

1. $y = \sqrt{x}$, $y = 0$ and $x = 4$, around the y-axis.



$y = 0$ is the x-axis

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right.

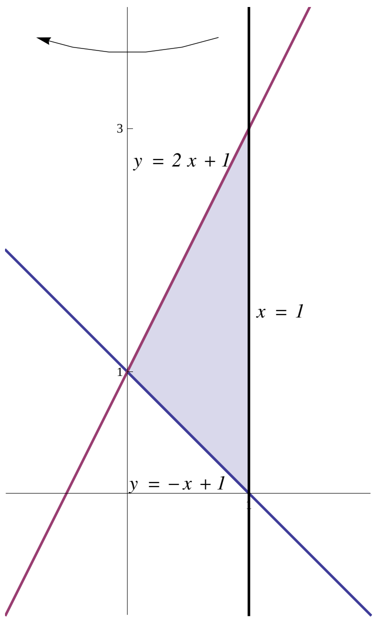
$x = 4$ is a straight line parallel to the y-axis and passing through $(4, 0)$.

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^4 x\sqrt{x}\ dx = 2\pi \int_0^4 x^{\frac{3}{2}}\ dx$$

$$\text{Volume} = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = 2\pi\ \frac{2}{5}\ (4)^{\frac{5}{2}} = 2\pi\ \frac{2}{5}\ (32) = \frac{128\pi}{5}$$

2. $x + y = 1$, $x = 1$ and $y = 2x + 1$, around the y-axis .



$y = -x + 1$ is a straight line with slope -1 and passing through $(0, 1)$.

$y = 2x + 1$ is a straight line with slope 2 and passing through $(0, 1)$.

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$.

Point of intersection of $x = 1$ and $y = -x + 1$ is $(1, 0)$.

Point of intersection of $x = 1$ and $y = 2x + 1$ is $(1, 3)$.

Point of intersection of $y = -x + 1$ and $y = 2x + 1$:

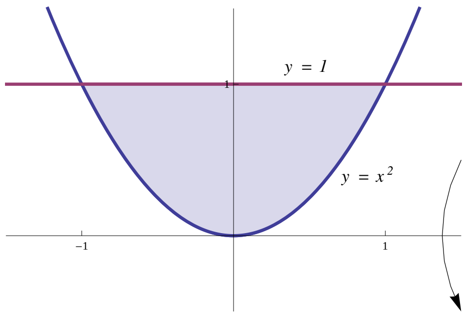
$2x + 1 = -x + 1 \ \Rightarrow\ 3x = 0 \ \Rightarrow\ x = 0.$

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 x[(2x+1) - (-x+1)]\ dx = 2\pi \int_0^1 x(3x)\ dx = 2\pi \int_0^1 3x^2\ dx$$

$$\text{Volume} = 2\pi \left[x^3\right]_0^1 = 2\pi[1 - 0] = 2\pi$$

3. $y = x^2$ and $y = 1$, around the x-axis.



$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = 1$ is a straight line parallel to the x-axis and passing through $(0, 1)$.

Since the bounded region is symmetric with respect to the y-axis, consider the right half of the parabola $y = x^2$ which is $x = \sqrt{y}$.

Using Cylindrical shells method

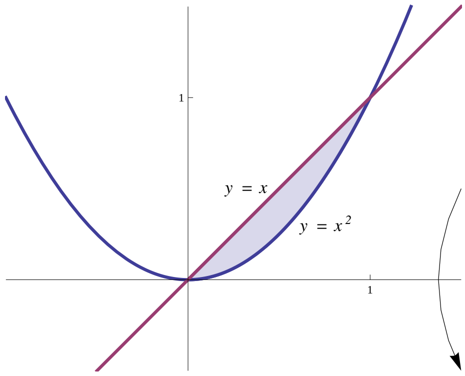
$$\text{Volume} = 2\left(2\pi \int_0^1 y\sqrt{y}\ dy\right) = 4\pi \int_0^1 y^{\frac{3}{2}}\ dy$$

$$\text{Volume} = 4\pi \left[\frac{2}{5} y^{\frac{5}{2}}\right]_0^1 = 4\pi\left(\frac{2}{5} - 0\right) = \frac{8\pi}{5}$$

4. $y = x^2$ and $y = x$, around the x-axis.

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = x$ is a straigh line passing through the origin with solpe 1.



Consider $x = \sqrt{y}$ which is the right half of the parabola $y = x^2$.

Points of intersection of $x = \sqrt{y}$ and $x = y$:

$y = \sqrt{y} \;\Rightarrow\; y^2 = y \;\Rightarrow\; y^2 - y = 0 \;\Rightarrow\; y(y-1) = 0 \;\Rightarrow\; y = 0 \;,\; y = 1$

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 y\left(\sqrt{y} - y\right)\, dy = 2\pi \int_0^1 \left(y^{\frac{3}{2}} - y^2\right)\, dy$$

$$\text{Volume} = 2\pi \left[\frac{2}{5}y^{\frac{5}{2}} - \frac{y^3}{3}\right]_0^1 = 2\pi \left[\left(\frac{2}{5} - \frac{1}{3}\right) - (0-0)\right]$$

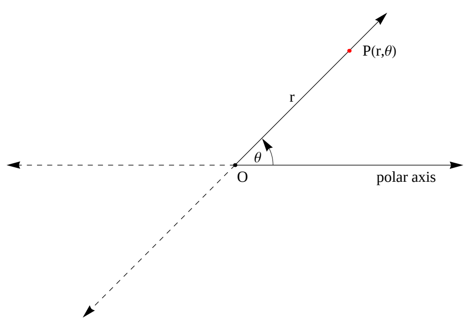
$$\text{Volume} = 2\pi \left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi \left(\frac{6-5}{15}\right) = \frac{2\pi}{15}$$

4. Polar Coordinates and Applications

4.1 Polar coordinates system :
In the recatangular coordinates system the ordered pair (a, b) represents a point, where "a" is the x-coordinat and "b" is the y-coordinate.

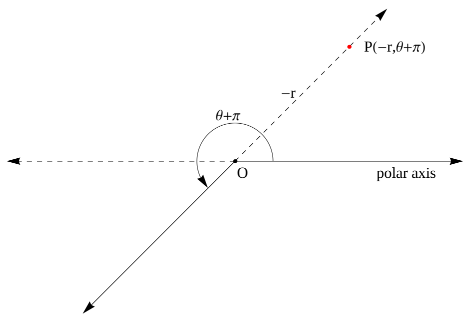
The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system , and the **polar axis** is the directed half-line (the non-negative part of the x-axis).

If P is any point in the plane different from the origin, then its polar coordinates consists of two components r and θ , where r is the distance between P and the pole O , and θ is the measure of the angle determined by the polar axis and OP.

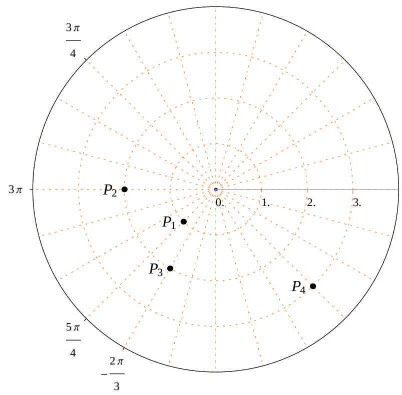


The meaning of polar coordinates (r, θ) can be extended to the case in which r is negative by considering the points (r, θ) and $(-r, \theta)$ lying on the same line through O and at a same distance $|r|$ from O but in opposite directions.

Remark : In this case the representation of a point using polar coordinates is not unique, for instance if $P(r, \theta)$ then other possible represenations are
$(-r, \pi + \theta)$, $(-r, \theta - \pi)$ $(r, \theta - 2\pi)$ and $(r, \theta \pm 2n\pi)$ where $n \in \mathbb{N}$.



Example 1: Plot the points whose polar coordinates are given :
$P_1\left(1, \dfrac{5\pi}{4}\right)$, $P_2(2, 3\pi)$, $P_3\left(2, -\dfrac{2\pi}{3}\right)$ and $P_4\left(-3, \dfrac{3\pi}{4}\right)$.
Solution :



Example 2: Write other polar reprsentations of the point $\left(1, \dfrac{\pi}{4}\right)$.
Solution :

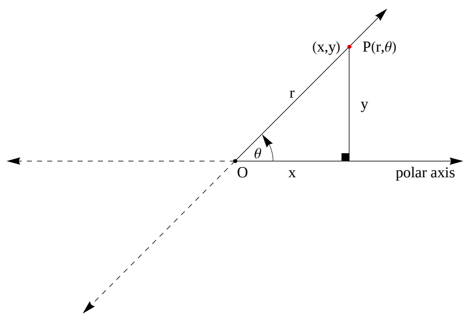
$$\left(-1, \frac{\pi}{4} + \pi\right) = \left(-1, \frac{5\pi}{4}\right).$$

$$\left(-1, \frac{\pi}{4} - \pi\right) = \left(-1, -\frac{3\pi}{4}\right)$$

$$\left(1, \frac{\pi}{4} - 2\pi\right) = \left(1, -\frac{7\pi}{4}\right)$$

$$\left(1, \frac{\pi}{4} + 3\pi\right) = \left(1, \frac{13\pi}{4}\right)$$

4.2 Relationship with Cartesian coordinates :



From the above figure , the relationship between the polar and cartesian coordinates is given by the formulas :

$\cos\theta = \dfrac{x}{r} \implies x = r\cos\theta$

$\sin\theta = \dfrac{y}{r} \implies y = r\sin\theta$

$r^2 = x^2 + y^2 \implies r = \sqrt{x^2 + y^2}$

$\tan\theta = \dfrac{y}{x} \implies \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ where $x \neq 0$.

Examples :

1. Convert the point $\left(2, \dfrac{\pi}{3}\right)$ from polar to Cartesian coordinates.
2. Convert the point $(1, 1)$ from Cartesian to polar coordinates.

Solution :

1. The point $\left(2, \dfrac{\pi}{3}\right)$ is written in polar coordinates where $r = 2$ and $\theta = \dfrac{\pi}{3}$

   $x = r\cos\theta = 2\cos\left(\dfrac{\pi}{3}\right) = 2 \times \dfrac{1}{2} = 1.$

   $y = r\sin\theta = 2\sin\left(\dfrac{\pi}{3}\right) = 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3}.$

   The Cartesian coordinates of the point $\left(2, \dfrac{\pi}{3}\right)$ is $\left(1, \sqrt{3}\right)$.

2. The point $(1, 1)$ is written in Cartesian coordinates where $x = 1$ and $y = 1$

   $r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

   $\tan\theta = \dfrac{y}{x} = \dfrac{1}{1} = 1 \implies \theta = \tan^{-1}(1) = \dfrac{\pi}{4}$

   The polar coordinates of the point $(1, 1)$ is $\left(\sqrt{2}, \dfrac{\pi}{4}\right)$

4.3 Polar curves:
A polar curve is an equation of r and θ of the form $r = r(\theta)$ or $r = f(\theta)$ where $\theta_1 \leq \theta \leq \theta_2$.

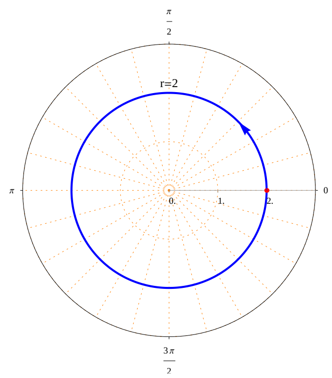
This section focuses on the circles centered at the origin and of radius $a > 0$. The polar curve $r = a$ where $a > 0$ represents a circle with center $(0, 0)$ and its radius equals a.

Examples : Sketch the following polar curves :

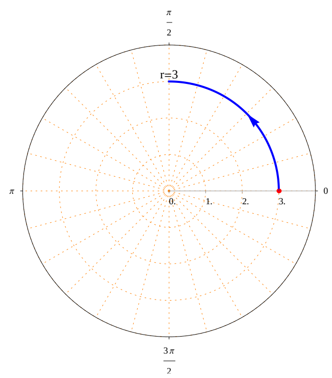
1. $r = 2$ where $0 \leq \theta \leq 2\pi$.
2. $r = 3$ where $0 \leq \theta \leq \dfrac{\pi}{2}$

Solution :

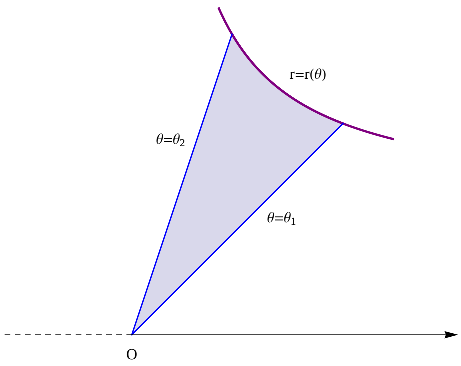
1. $r = 2$ where $0 \leq \theta \leq 2\pi$ represents a whole circle centered at $(0, 0)$ and its radius is 2.



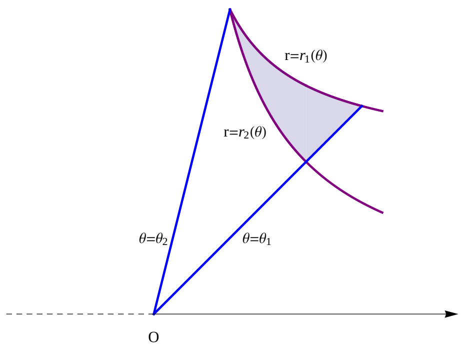
2. $r = 3$ where $0 \leq \theta \leq \dfrac{\pi}{2}$ represents the first quarter of a circle centered at $(0, 0)$ and its radius is 3.



4.4 Area with polar coordinates :

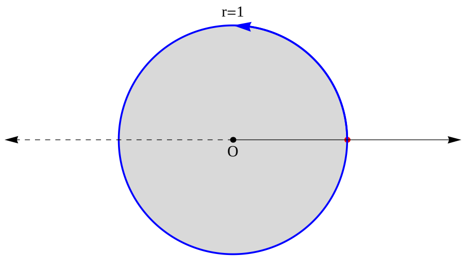


The area of the region bounded by the graph of $r = r(\theta)$, and the two lines $\theta = \theta_1$, $\theta = \theta_2$ is given by the formula
Area $= \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \; d\theta$



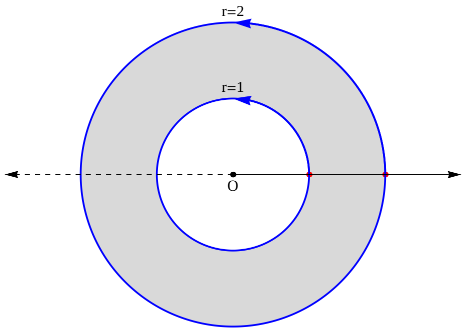
The area of the region bounded by the graphs of $r_1 = r_1(\theta)$, $r_2 = r_2(\theta)$ and the two lines $\theta = \theta_1$, $\theta = \theta_2$ is given by the formula
Area $= \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} \left([r_1(\theta)]^2 - [r_2(\theta)]^2\right) \; d\theta$

Example 1 : Find the area of the region inside the polar curve $r = 1$.
Solution : $r = 1$ is a whole circle centered at $(0, 0)$ and its radius is 1.



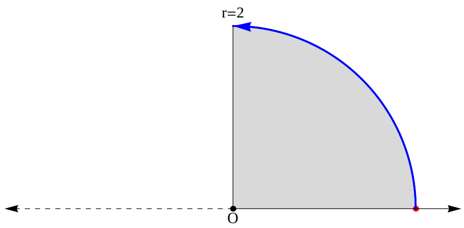
Area $= \frac{1}{2}\int_0^{2\pi} (1)^2 \ d\theta = \frac{1}{2}\int_0^{2\pi} 1 \ d\theta$
$= \frac{1}{2} \left[\theta\right]_0^{2\pi} = \frac{1}{2} \ [2\pi - 0] = \frac{1}{2} \times 2\pi = \pi$

Example 2 : Find the area of the region inside the polar curve $r = 2$ and outside the polar curve $r = 1$.
Solution : $r = 1$ is a whole circle centered at $(0, 0)$ and its radius is 1.
$r = 2$ is a whole circle centered at $(0, 0)$ and its radius is 2.



Area $= \frac{1}{2}\int_0^{2\pi} \left[(2)^2 - (1)^2\right] \ d\theta = \frac{1}{2}\int_0^{2\pi} (4-1) \ d\theta = \frac{1}{2}\int_0^{2\pi} 3 \ d\theta$
$= \frac{1}{2} \left[3\theta\right]_0^{2\pi} = \frac{1}{2} \ [3 \times 2\pi - 0] = \frac{1}{2} \times 6\pi = 3\pi$

Example 3 : Find the area of the region inside the polar curve $r = 2$ and at the first quadrant.
Solution : $r = 2$ is a circle centered at $(0, 0)$ and its radius is 2.
The region in the first quadrant means that it is bounded by the two lines $\theta = 0$ and $\theta = \frac{\pi}{2}$



Area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2 \ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4 \ d\theta$
$= \frac{1}{2} \left[4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2} \ [4 \times \frac{\pi}{2} - 0] = \frac{1}{2} \times 2\pi = \pi$

CHAPTER SIX

PARTIAL DERIVATIVES

1. Functions of several variables

2. Partial derivatives

3. Chain Rules

4. Implicit differentiation

1. Functions of several variables

1.1 Functions of two variables :
Definition: A function of two variables is a rule that assigns an ordered pair (x, y) (in the domain of the function) to a real number w.

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow w \end{aligned}$$

Example :

$f(x, y) = \dfrac{y}{x^2 + y^2}$ is a function of two variables x and y

$f(3, 1) = \dfrac{1}{3^2 + 1^2} = \dfrac{1}{10}.$

Note that $f(x, y)$ takes $(3, 1) \in \mathbb{R}^2$ to $\dfrac{1}{10} \in \mathbb{R}$

1.2 Functions of three variables :
Definition: A function of three variables is a rule that assigns an ordered triple (x, y, z) (in the domain of the function) to a real number w .

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longrightarrow w \end{aligned}$$

Example :

$f(x, y, z) = \dfrac{z}{x + y^2 + 3}$ is a function of three variables x , y and z

$f(1, -2, 4) = \dfrac{4}{1 + (-2)^2 + 3} = \dfrac{4}{8} = \dfrac{1}{2}.$

Note that $f(x, y, z)$ takes $(1, -2, 4) \in \mathbb{R}^3$ to $\dfrac{1}{2} \in \mathbb{R}$

2. Partial derivatives

2.1 Partial derivatives of a function of two variables :
If $w = f(x, y)$ is a function of two variables, then :

1. The partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y as a constant .
2. The partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x as a constant .

Example 1: Calculate f_x and f_y of the functiuon $f(x, y) = x^2y^3 + xy\ln(x + y)$
Solution:

1. $f_x = \dfrac{\partial}{\partial x}\left(x^2y^3 + xy\ln(x + y)\right)$

   $$f_x = (2x)y^3 + \left[(1)y\ln(x + y) + xy\frac{1}{x + y}\right] = 2xy^3 + y\ln(x + y) + \frac{xy}{x + y}$$

2. $f_y = \dfrac{\partial}{\partial y}\left(x^2y^3 + xy\ln(x + y)\right)$

   $$f_y = x^2(3y^2) + \left[x(1)\ln(x + y) + xy\frac{1}{x + y}\right] = 3x^2y^2 + x\ln(x + y) + \frac{xy}{x + y}$$

Example 2: Calculate f_x and f_y of the functiuon $f(x, y) = \dfrac{x + y^2}{x + y}$
Solution:

1. $f_x = \dfrac{\partial f}{\partial x} = \dfrac{(1 + 0)(x + y) - (x + y^2)(1 + 0)}{(x + y)^2} = \dfrac{x + y - (x + y^2)}{(x + y^2)}$

   $$f_x = \frac{x + y - x - y^2}{(x + y)^2} = \frac{y - y^2}{(x + y)^2}$$

2. $f_y = \dfrac{\partial f}{\partial y} = \dfrac{(0 + 2y)(x + y) - (x + y^2)(0 + 1)}{(x + y)^2} = \dfrac{2y(x + y) - (x + y^2)}{(x + y)^2}$

   $$f_y = \frac{2xy + 2y^2 - x - y^2}{(x + y)^2} = \frac{2xy - x + y^2}{(x + y)^2}$$

2.2 Partial derivatives of a function of three variables :
If $w = f(x, y, z)$ is a function of three variables, then :

1. The partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y and z as constants .

2. The partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x and z as constants .

3. The partial derivative of f with respect to z is denoted by $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial w}{\partial z}$, f_z or w_z , and it is calculated by applying the rules of differentiation to z and regarding x and y as constants .

Example : If $f(x, y, z) = 2z^3x - 4(x^2 + y^2)z$, then calculate f_x , f_y and f_z at $(0, 1, 2)$.
Solution :

1. $f_x = \dfrac{\partial}{\partial x}\big(2z^3x - 4(x^2 + y^2)z\big) = 2z^3 - 4(2x)z = 2z^3 - 8xz$

   $f_x(0, 1, 2) = 2\ (2^3) - 8(0)(2) = 16$

2. $f_y = \dfrac{\partial}{\partial y}\big(2z^3x - 4(x^2 + y^2)z\big) = 0 - 4(0 + 2y)z = -8yz$

   $f_y(0, 1, 2) = -8(1)(2) = -16$

3. $f_z = \dfrac{\partial}{\partial z}\big(2z^3x - 4(x^2 + y^2)z\big) = 6z^2x - 4(x^2 + y^2)$

   $f_z(0, 1, 2) = 6(2^2)(0) - 4(0^2 + 1^2) = -4$

2.3 Second partial derivatives :
If $w = f(x, y)$ is a function of two variables , then :

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}(f_x) = f_{xx}$.

2. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}(f_y) = f_{yy}$.

3. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}(f_y) = f_{yx}$.

4. $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}(f_x) = f_{xy}$.

Note : Second partial derivatives of a function of three variables are defined in a same manner.

Theorem : Let $f(x,y)$ be a function of two variables. If f , f_x , f_y , f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Note : If $f(x,y,z)$ is a function of three variables and f has continuous second partial derivatives, then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

Example 1: Let $f(x,y) = x^3y + xy^2 \sin(x+y)$, calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$

Solution :

$$f_x = 3x^2y + y^2 \sin(x+y) + xy^2 \cos(x+y)$$

$$f_y = x^3 + 2xy \sin(x+y) + xy^2 \cos(x+y)$$

$$f_{xy} = 3x^2 + 2y \sin(x+y) + y^2 \cos(x+y) + 2xy \cos(x+y) - xy^2 \sin(x+y)$$

$$f_{yx} = 3x^2 + 2y \sin(x+y) + 2xy \cos(x+y) + y^2 \cos(x+y) - xy^2 \sin(x+y)$$

Note : $f_{xy} = f_{yx}$ according to the theorem .

Example 2: Let $f(x,y,z) = x^3y^2z + xy \sin(y+z)$, calculate $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial z}$

Solution :

$$f_x = 3x^2y^2z + y \sin(y+z)$$

$$f_z = x^3y^2 + xy \cos(y+z)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = 6x^2yz + \sin(y+z) + y \cos(y+z)$$

$$\frac{\partial^2 f}{\partial x \partial z} = f_{zx} = 3x^2y^2 + y \cos(y+z)$$

Example 3: Let $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$, Show that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$

Solution :

$$f_x = 0 - 3z(2x) = -6xz$$

$$f_y = 0 - 3z(2y) = -6yz$$

$$f_z = 6z^2 - 3(x^2 + y^2)$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = -6z$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = -6z$$

$$\frac{\partial^2 f}{\partial z^2} = f_{zz} = 12z$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$$

3. Chain Rules

Theorem (Chain Rules):

1. If $w = f(x,y)$ and $x = g(t)$, $y = h(t)$, such that f , g and h are differentiable then

   $$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\,\frac{dx}{dt} + \frac{\partial w}{\partial y}\,\frac{dy}{dt}$$

2. If $w = f(x,y)$ and $x = g(t,s)$, $y = h(t,s)$, such that f , g and h are differentiable then

   $$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t}$$

   $$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}$$

3. If $w = f(x,y,z)$ and $x = g(t,s)$, $y = h(t,s)$, $z = k(t,s)$ such that f , g , h and k are differentiable then

   $$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial t}$$

   $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial s}$$

Example 1 : Let $f(x,y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution :

1. $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial s}$

   $\dfrac{\partial f}{\partial x} = y$, $\dfrac{\partial x}{\partial s} = 2st$

   $\dfrac{\partial f}{\partial y} = x + 2y$, $\dfrac{\partial y}{\partial s} = 1$

   $\dfrac{\partial f}{\partial s} = y\ (2st) + (x+2y)(1) = (s+t)2st + \left[s^2 t + 2(s+t)\right]$

   $= 2s^2 t + 2st^2 + s^2 t + 2s + 2t = 3s^2 t + 2st^2 + 2s + 2t$

2. $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial t}$

   $\dfrac{\partial f}{\partial x} = y$, $\dfrac{\partial x}{\partial t} = s^2$

   $\dfrac{\partial f}{\partial y} = x + 2y$, $\dfrac{\partial y}{\partial t} = 1$

$$\frac{\partial f}{\partial t} = ys^2 + (x+2y)(1) = (s+t)s^2 + s^2t + 2(s+t)$$

$$= s^3 + s^2t + s^2t + 2s + 2t = s^3 + 2s^2t + 2s + 2t$$

Example 2 : Let $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s+t$ and $z = \dfrac{s}{t}$, calculate $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution :

1. $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\ \frac{\partial z}{\partial s}$$

   $$\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)\ ,\ \frac{\partial x}{\partial s} = t$$

   $$\frac{\partial f}{\partial y} = x\cos(xy)\ ,\ \frac{\partial y}{\partial s} = 1$$

   $$\frac{\partial f}{\partial z} = -x\sin(xz)\ ,\ \frac{\partial z}{\partial s} = \frac{1}{t}$$

   $$\frac{\partial f}{\partial s} = t\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{1}{t}\right)(-x\sin(xz))$$

   $$\frac{\partial f}{\partial s} = t + ty\cos(xy) - tz\sin(xz) + x\cos(xy) - \frac{x\sin(xz)}{t}$$

2. $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\ \frac{\partial z}{\partial t}$$

   $$\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)\ ,\ \frac{\partial x}{\partial t} = s$$

   $$\frac{\partial f}{\partial y} = x\cos(xy)\ ,\ \frac{\partial y}{\partial t} = 1$$

   $$\frac{\partial f}{\partial z} = -x\sin(xz)\ ,\ \frac{\partial z}{\partial t} = \frac{-s}{t^2}$$

   $$\frac{\partial f}{\partial t} = s\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{-s}{t^2}\right)(-x\sin(xz))$$

   $$\frac{\partial f}{\partial t} = s + sy\cos(xy) - sz\sin(xz) + x\cos(xy) + \frac{sx\sin(xz)}{t^2}$$

4. Implicit differentiation

1. Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x say $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

2. Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable , then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 1 : Let $y^2 - xy + 3x^2 = 0$, find $\dfrac{dy}{dx}$.
Solution 1: Let $F(x, y) = x^2 - xy + 3x^2$ then $F(x, y) = 0$

$F_x = -y + 6x$ and $F_y = 2y - x$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y + 6x)}{2y - x} = \frac{y - 6x}{2y - x} \ .$$

Solution 2 : $y^2 - xy + 3x^2 = 0$

Differentiate both sides implicitly

$2yy' - (y + xy') + 6x = 0 \quad \Rightarrow \quad 2yy' - y - xy' + 6x = 0$

$\Rightarrow \quad 2yy' - xy' = y - 6x \quad \Rightarrow \quad (2y - x)y' = y - 6x$

$$\Rightarrow \quad \frac{dy}{dx} = y' = \frac{y - 6x}{2y - x}$$

Example 2 : Let $F(x, y, z) = x^2y + z^2 + \sin(xyz) = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution :

$F_x = 2xy + yz\cos(xyz)$

$F_y = x^2 + xz\cos(xyz)$

$F_z = 2z + xy\cos(xyz)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}$$

CHAPTER SEVEN

DIFFERENTIAL EQUATIONS

1. Definition of a differential equation

2. Separable Differential equations

3. First-order linear differential equations

1. Definition of a differential equation

Definition : An equation that involves x , y , y' , y'' , $\cdots$, $y^{(n)}$ for a function $y(x)$ with n^{th} derivative $y^{(n)}$ of y with respect to x is an ordinary differential Equation of order n.

Examples :

1. $y' = x^2 + 5$ is a differential equation of order 1.
2. $y'' + x\,(y')^4 - y = x$ is a differential equation of order 2
3. $\left(y^{(4)}\right)^3 + x^2\;y'' = 2x$ is a differnetial equation of order 4

$y = y(x)$ is called a **solution** of a differential equation if $y = y(x)$ satisfies that differential equation.
Consider the differential equation $y' = 6x + 4$, then $y = 3x^2 + 4x + c$ is the **general solution** of that differential equation.

If an **initial condition** was added to the differential equation to assign a certain vallue for c then $y = y(x)$ is called the **particular solution** of the differential equation .
Consider the differential equation $y' = 6x+4$ with the initial condition $y(0) = 2$, $y = 3x^2 + 4x + c$ is the general solution of the differential equation ,
$y(0) = 2 \;\Rightarrow\; 3(0)^2 + 4 \times 0 + c = 2 \;\Rightarrow\; c = 2$, hence $y = 3x^2 + 4x + 2$ is the particular solution of the differential equation.

2. Separable Differential equations

The separable differential equation has the form $M(x) + N(y)\ y' = 0$.
where $M(x)$ and $N(y)$ are continuous functions.

To solve the separable differential equation :

1. Write it as $M(x)\ dx + N(y)\ dy = 0 \implies N(y)\ dy = -M(x)\ dx$.
2. Integrate the left-hand side with respect to y and the right-hand side with respect to x

   $$\int N(y)\ dy = -\int M(x)\ dx$$

Example 1 : Solve the differential equation $y' + y^3 e^x = 0$.
Solution :

$$y' + y^3 e^x = 0 \implies \frac{dy}{dx} = -y^3 e^x \ ,$$

$$\implies -\frac{1}{y^3}\ dy = e^x\ dx \implies -y^{-3}\ dy = e^x\ dx$$

$$\implies -\int y^{-3}\ dy = \int e^x\ dx \implies -\frac{y^{-2}}{-2} = e^x + c$$

$$\implies \frac{1}{2y^2} = e^x + c \implies \frac{1}{y^2} = 2\left(e^x + c\right)$$

$$\implies y^2 = \frac{1}{2\left(e^x + c\right)} \implies y = \sqrt{\frac{1}{2\left(e^x + c\right)}}$$

Example 2 : Solve the differential equation $\dfrac{dy}{dx} = y^2 e^x$, $y(0) = 1$.
Solution :

$$\frac{dy}{dx} = y^2 e^x \implies \frac{1}{y^2}\ dy = e^x\ dx$$

$$\implies y^{-2}\ dy = e^x\ dx \implies \int y^{-2}\ dy = \int e^x\ dx$$

$$\implies \frac{y^{-1}}{-1} = e^x + c \implies y = \frac{-1}{e^x + c}$$

Using the initial condition $y(0) = 1 \implies 1 = \dfrac{-1}{e^0 + c}$

$$\implies 1 = \frac{-1}{1 + c} \implies 1 + c = -1 \implies c = -2$$

The particular solution is $y = \dfrac{-1}{e^x - 2}$

Example 3 : Solve the differential equation $dy - \sin x(1+y^2)dx = 0$.
Solution :

$$dy - \sin x(1+y^2)dx = 0 \implies dy = \sin x(1+y^2)dx$$

$$\implies \frac{1}{1+y^2}\ dy = \sin x\ dx \implies \int \frac{1}{1+y^2}\ dy = \int \sin x\ dx$$

$$\implies \tan^{-1} y = -\cos x + c \implies y = \tan(-\cos x + c)$$

Example 4 : Solve the differential equation $e^{-y}\sin x - y'\cos^2 x = 0$.
Solution :

$$e^{-y}\sin x - y'\cos^2 x = 0 \implies -\cos^2 x\frac{dy}{dx} = -e^{-y}\sin x$$

$$\implies \frac{1}{e^{-y}}\ dy = \frac{-\sin x}{-\cos^2 x}\ dx \implies e^y\ dy = \frac{1}{\cos x}\ \frac{\sin x}{\cos x}\ dx$$

$$\implies e^y\ dy = \sec x \tan x\ dx \implies \int e^y\ dy = \int \sec x \tan x\ dx$$

$$\implies e^y = \sec x + c \implies y = \ln|\sec x + c|$$

Example 5 : Solve the differential equation $y' = 1 - y + x^2 - yx^2$.
Solution :

$$y' = 1 - y + x^2 - yx^2 \implies \frac{dy}{dx} = 1 - y + x^2(1-y)$$

$$\implies \frac{dy}{dx} = (1-y)(1+x^2) \implies \frac{1}{1-y}\ dy = (1+x^2)\ dx$$

$$\implies \int \frac{1}{1-y}\ dy = \int (1+x^2)\ dx \implies -\int \frac{-1}{1-y}\ dy = \int (1+x^2)\ dx$$

$$\implies -\ln|1-y| = x + \frac{x^3}{3} + c \implies \ln|1-y| = -x - \frac{x^3}{3} - c$$

$$\implies 1 - y = e^{-x-\frac{x^3}{3}-c} \implies y = 1 - e^{-x-\frac{x^3}{3}-c}$$

3. First-order linear differential equations

The first-order linear differential equation has the form $y' + P(x)\ y = Q(x)$, where $P(x)$ and $Q(x)$ are continuous functions of x

To solve the first-order linear differential equation :

1. Compute the integrating factor $u(x) = e^{\int P(x)\ dx}$
2. The general solution of the first-order linear differential equation is

   $$y(x) = \frac{1}{u(x)} \int u(x)\ Q(x)\ dx$$

Example 1 : Solve the differential equation $x\dfrac{dy}{dx} + y = x^2 + 1$.
Solution :

$$x\frac{dy}{dx} + y = x^2 + 1 \quad \Longrightarrow \quad y' + \left(\frac{1}{x}\right) y = \frac{x^2+1}{x}$$

$$\Longrightarrow \quad y' + \left(\frac{1}{x}\right) y = x + \frac{1}{x}$$

$P(x) = \dfrac{1}{x}$ and $Q(x) = x + \dfrac{1}{x}$

The integrating factor is $u(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

The general solution is $y = \dfrac{1}{x} \displaystyle\int x \left(x + \frac{1}{x}\right)\ dx$

$$y = \frac{1}{x} \int (x^2 + 1)\ dx = \frac{1}{x}\left(\frac{x^3}{3} + x + c\right) = \frac{x^2}{3} + 1 + \frac{c}{x}$$

Example 2 : Solve the differential equation $y' - \dfrac{2}{x}y = x^2 e^x$, $y(1) = e$.
Solution :

$P(x) = -\dfrac{2}{x}$ and $Q(x) = x^2 e^x$

The integrating factor is

$u(x) = e^{\int -\frac{2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

The general solution is $y = \dfrac{1}{x^{-2}} \displaystyle\int x^{-2} x^2 e^x dx$

$$y = x^2 \int e^x dx = x^2(e^x + c) = x^2 e^x + cx^2$$

Using the initial condition $y(1) = e$

$$y(1) = e \implies e = (1)^2 e^1 + c\ (1)^2 \implies e = e + c \implies c = 0$$

The particular solution is $y = x^2 e^x$

Example 3 : Solve the differential equation $y' + y = \cos(e^x)$
Solution :

$P(x) = 1$ and $Q(x) = \cos(e^x)$

The integrating factor is $u(x) = e^{\int 1\ dx} = e^x$

The general solution is $y = \dfrac{1}{e^x} \displaystyle\int e^x \cos(e^x)\ dx$

$$y = e^{-x} \int \cos(e^x) e^x\ dx = e^{-x} \left(\sin(e^x) + c\right) = e^{-x} \sin(e^x) + c e^{-x}$$

Example 4 : Solve the differential equation $xy' - 3y = x^2$
Solution :

$$xy' - 3y = x^2 \implies y' - \frac{3}{x} y = x$$

$P(x) = -\dfrac{3}{x}$ and $Q(x) = x$

The integrating factor is

$$u(x) = e^{\int -\frac{3}{x} dx} = e^{-3 \int \frac{1}{x} dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$$

The general solution is $y = \dfrac{1}{x^{-3}} \displaystyle\int x^{-3} x\ dx$

$$y = x^3 \int x^{-2} dx = x^3 \left(\frac{x^{-1}}{-1} + c \right)$$

$$y = x^3 \left(-\frac{1}{x} + c \right) = -x^2 + c x^3$$