King Saud University: Mathematics Department M-254
Semester I (1st Midterm Exam Solution) 1440-1441 H
Maximum Marks=25 Time Allowed: 90 Mins.

Questions : (5 + 5 + 5 + 5 + 5)

Q1: Use Newton's method with $x_0 = 0$ to find the second approximation of the value of x that produces the point on the graph of $y = x^2$ that is closest to the point $(3, 2)$. What is the value of the point (x, y) on the graph of $y = x^2$.

Solution. The distance between an arbitrary point (x, x^2) on the graph of $y = x^2$ and the point $(1, 0)$ is

$$d(x) = \sqrt{(x-3)^2 + (x^2-2)^2} = \sqrt{x^4 - 3x^2 - 6x + 13}.$$

Because a derivative is needed to find the critical point of d, it is easier to work with the square of this function

$$F(x) = [d(x)]^2 = x^4 - 3x^2 - 6x + 13,$$

whose minimum will occur at the same value of x as the minimum of $d(x)$. To minimize $F(x)$, we need x so that

$$F'(x) = 4x^3 - 6x - 6 = 0, \quad \text{gives} \quad f(x) = 4x^3 - 6x - 6 \quad \text{and} \quad f'(x) = 12x^2 - 6.$$

Applying Newton's iterative formula to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{4x_n^3 - 6x_n - 6}{12x_n^2 - 6}.$$

Using the initial approximation $x_0 = 0$, we get

$$x_1 = x_0 - \frac{4x_0^3 - 6x_0 - 6}{12x_0^2 - 6} = -1.$$

Continue in the same manner, we get, $x_2 = -1/3$. So the point on the graph that is closest to $(3, 2)$ has the approximate coordinates $(-1/3, 1/9)$.

Q2: Show that the rate of convergence of the Newton's method at the root $x = 0$ of the equation $x^2e^x = 0$ is linear. Use quadratic convergence method to find second approximation to the root using $x_0 = 0.1$. Also, compute the absolute error.

Solution. Given $f(x) = x^2e^x$ and so $f'(x) = (x^2 + 2x)e^x$. Using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{(x_n^2 e^{x_n})}{(x_n^2 + 2x_n)e_n^x} = \frac{(x_n + x_n^2)}{(2 + x_n)}, \quad n \geq 0.$$

The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = \frac{(x_n + x_n^2)}{(2 + x_n)}.$$

Then

$$g(x) = \frac{(x + x^2)}{(2 + x)}, \quad g'(x) = \frac{(x^2 + 4x + 2)}{(2 + x)^2}, \quad g'(0) = \frac{1}{2} \neq 0.$$

Thus the method converges linearly to the given root.
The quadratic convergent method is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function. To find m, we do

$$f''(x) = (x^2 + 4x + 2)e^x, \quad \text{and} \quad f''(0) = 2 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{x_n^2 e^{x_n}}{(x_n^2 + 2x_n)e_n^x} = x_n - 2\frac{x_n^2}{(x_n^2 + 2x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 0.1$, we have

$$x_1 = x_0 - 2\frac{x_0^2}{(x_0^2 + 2x_0)} = 0.00476, \quad x_2 = x_1 - 2\frac{x_1^2}{(x_1^2 + 2x_1)} = 0.0000311,$$

the required two approximations. The possible absolute error is

$$|\alpha - x_2)| = |0.0 - 0.0000311| = 0.0000311.$$

Q3: Show that the secant method for finding approximation of the cubic root of a positive number N is

$$x_{n+1} = \frac{x_n x_{n-1}(x_n + x_{n-1}) + N}{x_n^2 + x_n x_{n-1} + x_{n-1}^2}, \quad n \geq 1.$$

Carry out the first two approximations for the cubic root of 27, using $x_0 = 2, x_1 = 2.5$ and also compute absolute error.

Solution. We shall compute $x = N^{1/3}$ by finding a positive root for the nonlinear equation

$$x^3 - N = 0,$$

where $N > 0$ is the number whose root is to be found. If $f(x) = 0$, then $x = \alpha = N^{1/3}$ is the exact zero of the function

$$f(x) = x^3 - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1.$$

Hence, assuming the initial estimates to the root, say, $x = x_0, x = x_1$, and by using the secant iterative formula, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^3 - N)}{(x_1^3 - N) - (x_0^3 - N)} = x_1 - \frac{(x_1 - x_0)(x_1^3 - N)}{(x_1 - x_0)(x_1^2 + x_1 x_0 + x_0^2)} = \frac{x_1 x_0(x_1 + x_0) + N}{x_1^2 + x_1 x_0 + x_0^2}.$$

In general, we have

$$x_{n+1} = \frac{x_n x_{n-1}(x_n + x_{n-1}) + N}{x_n^2 + x_n x_{n-1} + x_{n-1}^2}, \qquad n = 1, 2, \ldots,$$

the secant formula for approximation of the square root of number N. Now using this formula for approximation of the square root of $N = 27$, taking $x_0 = 2$ and $x_1 = 2.5$, we have

$$x_2 = 3.2459, \qquad \text{and} \qquad x_3 = 2.9568.$$

Hence

$$\text{Absolute Error} = |27^{1/3} - x_3| = |3 - 2.9568| = 0.0431,$$

is the possible absolute error.

Q4: Find the first approximation of the point of intersection of the circle $x^2 + y^2 = 1$ and the ellipse $\frac{1}{3}x^2 + \frac{1}{2}y^2 = 1$ using Newton's method, starting with initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Solution. We are given the nonlinear system

$$\begin{array}{rrcl} x^2 & +y^2 & = & 1 \\ \frac{1}{3}x^2 & +\frac{1}{2}y^2 & = & 1 \end{array}$$

and it gives the functions and the first partial derivatives as follows:

$$\begin{array}{rcl} f_1(x,y) & = & x^2 + y^2 - 1, \quad f_{1x} = 2x, \quad f_{1y} = 2y, \\ f_1(x,y) & = & \frac{1}{3}x^2 + \frac{1}{2}y^2 - 1, \quad f_{2x} = \frac{2}{3}x, \quad f_{2y} = y. \end{array}$$

At the given initial approximation $x_0 = 1$ and $y_0 = 1$, we get

$$\begin{array}{rclrclrcl} f_1(1,1) & = & 1, & \dfrac{\partial f_1}{\partial x} = f_{1x} & = & 2, & \dfrac{\partial f_1}{\partial y} = f_{1y} & = & 2, \\ f_2(1,1) & = & -\frac{1}{6}, & \dfrac{\partial f_1}{\partial x} = f_{2x} & = & \frac{2}{3}, & \dfrac{\partial f_2}{\partial y} = f_{2y} & = & 1. \end{array}$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as follows:

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ \frac{2}{3} & 1 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{2/3} \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & 2 \end{pmatrix}.$$

Substituting all these values, we get the first approximation as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2/3} \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} -1 \\ 2.5 \end{pmatrix}.$$

Q5: Convert the equation $2^x - 5x + 1 = 0$ to the fixed-point problem

$$x = \frac{1}{1+c}\left(cx + \frac{2^x+1}{5}\right),$$

with c a constant. Find a value of c to ensure rapid convergence of the following scheme near $x = 0.1$

$$x_{n+1} = \frac{1}{1+c}\left(cx_n + \frac{2^{x_n}+1}{5}\right), \qquad n \geq 0.$$

Compute the third iterates, starting with $x_0 = 0.1$.

Solution. Given $2^x - 5x + 1 = 0$ and it can be written as for any c

$$x(c - c + 1) = \frac{2^x+1}{5} \quad \text{or} \quad x(c+1) - xc = \frac{2^x+1}{5} \quad \text{or} \quad x(c+1) = xc + \frac{2^x+1}{5}.$$

From this we have

$$x = \frac{1}{1+c}\left(cx + \frac{2^x+1}{5}\right) = g(x),$$

and it gives the iterative scheme

$$x_{n+1} = \frac{1}{1+c}\left(cx_n + \frac{2^{x_n}+1}{5}\right) = g(x_n), \qquad n \geq 0.$$

For guaranteed the convergence will be rapid if

$$g'(x) = 0, \quad \text{gives} \quad c = -\frac{2^x \ln 2}{5}.$$

Thus, $c = g'(0.1) = -\frac{2^x \ln 2}{5} = -0.1486$. Now to find third iterates when $x_0 = 0.5$

$$\begin{aligned} x_1 &= \frac{1}{1+c}\left(cx_0 + \frac{2^{x_0}+1}{5}\right) = 0.4692 \\ x_2 &= \frac{1}{1+c}\left(cx_1 + \frac{2^{x_1}+1}{5}\right) = 0.4782 \\ x_3 &= \frac{1}{1+c}\left(cx_2 + \frac{2^{x_2}+1}{5}\right) = 0.4787, \end{aligned}$$

the required approximations at the value of $c = -0.1486$.