King Saud University
Department of Mathematics

Mid Term Exam **280-Math** **1Semester (1441/1442)H**

Question1(6). (a) Decide whether the set $E=\{\sqrt{n+1}-\sqrt{n}\ \ ,\ n\in\mathbb{N}\}$ is bounded or not.

(b) Determine $\sup E$ and $\inf E$ (without using the limit).

Question2(6). Find $\lim\limits_{n\to\infty} x_n$ or show that it is divergent if

(a) $$x_n=\frac{n^2}{\sqrt{n^6+1}}+\frac{n^2}{\sqrt{n^6+2}}+\cdots+\frac{n^2}{\sqrt{n^6+n}}$$

(b) $$x_n=\frac{1}{1^2+1}+\frac{1}{2^2+2}+\cdots+\frac{1}{n^2+n}$$

Question3 (6). (a) Use appropriate method to decide whether the sequence $x_n=\sum\limits_{k=1}^{n}\frac{3k^2+2k}{2^k}$

converges or diverges.

(b) Decide whether the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{(1)(3)(5)....(2n-1)}{3^n n!}$

converges (absolutely or conditionally) or diverges

Question4(6). (a) Show that the equation $2^{x^2+x+1}-x^3-20x+2=0$ has at least two real solutions.

(b) Show that if $f(x):[0,1]\to[0,1]$ and $f(x)$ is continuous on $[0,1]$, then

$\exists c\in[0,1]$ such that $f(c)=2c$

Question5(6). (a) Explain whether the following function is bounded or not

$$f(x)=\frac{(1+\sqrt{x})^{2n}-(1-\sqrt{x})^{2n}}{(1+\sqrt{x^2+1})^n}$$ on the interval $[0,2n]$.

(b) Decide whether the function $f(x)=e^{-\frac{1}{x}}$ is uniformly continuous on the interval $(0,1)$.

Solutions

Question1(6). (a) Let $x_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

It is obvious that $0 < \dfrac{1}{\sqrt{n+1}+\sqrt{n}} < \dfrac{1}{\sqrt{n}} \le 1$ or $0 < x_n < 1 \ \forall n \in \mathbb{N}$

So the set E is bounded .

(b) It is clear that $x_{n+1} = \dfrac{1}{\sqrt{n+2}+\sqrt{n+1}} < \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = x_n$

The conclusion one can make is the following: $x \le x_1 \quad \forall x \in E$

Since $x_1 \in E$, we get $\sup E = x_1 = \dfrac{1}{\sqrt{2}+1} = \sqrt{2} - 1$.

Now we will prove that $\inf E = 0$.

First we have $0 < x_n < 1 \ \forall n \in \mathbb{N}$. Hence 0 is a lower bound of the set E. Let $0 < x$.

Using Ar p for the positive number x^2 there is $n \in \mathbb{N}$ such that $\dfrac{1}{n} < x^2$.

It follows that $\dfrac{1}{\sqrt{n+1}+\sqrt{n}} < \dfrac{1}{\sqrt{n}} < x$. Since the number $\dfrac{1}{\sqrt{n+1}+\sqrt{n}} \in E$, we conclude that the number x is not an lower bound of the set E. thus 0 is the largest lower bound of E. it means that $\inf E = 0$.

Question2(6). (a) If $x_n = \dfrac{n^2}{\sqrt{n^6+1}} + \dfrac{n^2}{\sqrt{n^6+2}} + \cdots + \dfrac{n^2}{\sqrt{n^6+n}}$, then

$$\frac{1}{\sqrt{1+\frac{1}{n^5}}} = \frac{n\,n^2}{\sqrt{n^6+n}} \le x_n \le \frac{n\,n^2}{\sqrt{n^6+1}} < \frac{n\,n^2}{\sqrt{n^6}} = 1$$

Passing to the limit we get $\lim\limits_{n\to\infty} x_n = 1$ (by squeezing rule)

(b) $x_n = \dfrac{1}{1^2+1} + \dfrac{1}{2^2+2} + \cdots + \dfrac{1}{n^2+n}$

$= \dfrac{1}{1(1+1)} + \dfrac{1}{2(2+1)} + \cdots + \dfrac{1}{n(n+1)}$

$$=\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}$$

$$=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{n}-\frac{1}{n+1})$$

$=1-\frac{1}{n+1}$. So $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(1-\frac{1}{n+1})=1$

Question3 (6)**.** (a) note that the sequence $x_n = \sum_{k=1}^{n}\frac{3k^2+2k}{2^k}$ is the partial sum of the series

$$\sum_{n=1}^{\infty}\frac{3n^2+2n}{2^n}$$

Appling the Ratio test (or root test) we get: $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{2}<1$

Hence we get that the series converges and therefore the sequence x_n converges .

(b) Appling the Ratio test to the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(1)(3)(5)....(2n-1)}{3^n n!}$,

we have $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2n+1}{3(n+1)}=\frac{2}{3}<1$. So the series $\sum_{n=1}^{\infty} a_n$ converges. Therefore the given series absolutely converges, hence it converges.

Question4(6)**.** (a) Define $f(x)=2^{x^2+x+1}-x^3-20x+2$

Noting that $f(0)=4>0$ and $f(1)=-11<0$ we deduce that $\exists c_1\in(0,1)\ st\ f(c_1)=0$

Noting that $f(2)=2^7-8-40+2=82>0$ we deduce that $\exists c_2\in(1,2)\ st\ f(c_2)=0$

Thus the equation $2^{x^2+x+1}-x^3-20x+2=0$ has at least two real solutions.

(b) If $f(0)=0$, then the number 0 satisfies the requirement.

Now let $f(0)\neq 0$, then $f(0)>0$. Let $F(x)=f(x)-2x$.

We have $F(0)=f(0)>0$ and $F(1)=f(1)-2<0$.

The function $F(x)$ is continuous on $[0,1]$. Appling the MVT we deduce that $\exists c\in[0,1]$ such that

$$F(c)=0 \Rightarrow f(c)-2c=0 \Rightarrow f(c)=2c$$

Question5(6). (a) By properties of continuous functions we see that the function

$$f(x)=\frac{(1+\sqrt{x})^{2n}-(1-\sqrt{x})^{2n}}{(1+\sqrt{x^2+1})^n}$$ is continuous on the closed and bounded interval $[0,2n]$.

Using boundedness theorem we conclude that $f(x)$ is bounded on the interval $[0,2n]$.

(b) Define the function $g(x)=\begin{cases} e^{-\frac{1}{x}} & , \ x\in(0,1] \\ 0 & , \ x=0 \end{cases}$

Because $\lim_{x\to 0} g(x)=0$, the function $g(x)$ is continuous on the interval $[0,1]$.

Furthermore $g(x)\equiv f(x)$ on the interval $(0,1)$.

Using Continuous Extension Theorem we conclude that the function $f(x)=e^{-\frac{1}{x}}$ is uniformly continuous on the interval $(0,1)$.