KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS

Math 254 (SEMESTER 1, 1437-1438) FIRST MID-TERM EXAM

FULL MARKS: 25 TIME 90min

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**Question 1** [5 Marks]

Let $g \in C[a,b]$ with $g(x) \in [a,b]$ forall $x \in [a,b]$ and $g'(x)$ is a continuous function on $(a,b)$ such that $|g'(x)| \leq \lambda < 1$ for all $x \in (a,b)$. Show that

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|.$$

**Question 2** [5 Marks]

If $x = \alpha$ be a root of $f(x) = 0$ and $f'(\alpha) = f''(\alpha) = f'''(\alpha) = 0$ but $f^{(4)}(\alpha) \neq 0$. show that the rate of convergence of first modified Newton's method is at least quadratic.

**Question 3** [5 Marks]

Show that the nonlinear equation

$$x^3 - 13x^2 + 35x + 49 = 0$$

has a root at $x = 7$ and use a quadratic convergent iterative method to find its first two approximations using $x_0 = 9$.

**Question 4** [5 Marks]

Show that

$$\ln x - x + 1.5 = 0$$

has a solution in $[2,3]$ and then use the bisection method to compute the first three approximations. Also, compute an error bound and the absolute error for your approximation.

**Question 5** [5 Marks]

Find the value of $a$ and $b$ so that the rate of convergence of the iterative scheme

$$x_{n+1} = ax_n + \frac{bN}{x_n^2} \quad , n \geq 0$$

for computing the third root of a positive number $N$ becomes quadratic.

Q.1).

$\alpha = g(\alpha)$ , $x_n = g(x_{n-1})$

$|\alpha - x_n| = |g(\alpha) - g(x_{n-1})| = |g'(\eta)(\alpha - x_{n-1})|$ by mean value theorem $\alpha < \eta < x_{n-1}$

$= |g'(\eta)||\alpha - x_{n-1}|$

$$\left.\begin{aligned} |\alpha - x_n| &\le \lambda |\alpha - x_{n-1}| \\ |\alpha - x_{n-1}| &\le \lambda |\alpha - x_{n-2}| \\ &\vdots \\ |\alpha - x_1| &\le \lambda |\alpha - x_0| \end{aligned}\right\} \quad |\alpha - x_n| \le \lambda^n |\alpha - x_0| \quad (1)$$

$|\alpha - x_0| = |\alpha - x_1 + x_1 - x_0|$

$\le |\alpha - x_1| + |x_1 - x_0| \le \lambda |\alpha - x_0| + |x_1 - x_0|$

Then $(1-\lambda)|\alpha - x_0| \le |x_1 - x_0|$

$|\alpha - x_0| \le \dfrac{1}{1-\lambda} |x_1 - x_0|$ (2)

using (1) and (2) we obtain

$$|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0|$$

Q2) $\alpha$ is a root of order 4 $\Rightarrow f(x) = (x-\alpha)^4 h(x)$ , $h(\alpha) \neq 0$

First modified Newton : $g(x) = x - 4 \dfrac{f(x)}{f'(x)}$

$= x - 4 \dfrac{(x-\alpha)^4 h(x)}{4(x-\alpha)^3 h(x) + (x-\alpha)^4 h'(x)}$

$= x - 4 \dfrac{(x-\alpha) h(x)}{4 h(x) + (x-\alpha) h'(x)}$

$$g'(x) = 1 - 4\,\frac{[h(x)+(x-\alpha)h'(x)]\,.\,[4h(x)+(x-\alpha)h'(x)]}{[4h(x)+(x-\alpha)h'(x)]^2}$$

$$\frac{-(x-\alpha)h(x)\,[4h(x)+(x-\alpha)h'(x)]'}{}$$

$$g'(\alpha) = 1 - 4\,\frac{[h(\alpha)]\,.\,4h(\alpha)}{[4h(\alpha)]^2} = 1 - \frac{16\,h^2(\alpha)}{16\,h^2(\alpha)} = 0$$

Then the rate of convergence is at least quadratic

Q3)

$$f(x) = x^3 - 13x^2 + 35x + 49$$

$$f(7) = 343 - 637 + 245 + 49 = 0$$ Then $f$ has a root at $x = 7$

$$f'(x) = 3x^2 - 26x + 35$$

$$f'(7) = 147 - 182 + 35 = 0$$

$$f''(x) = 6x - 26$$

$$f''(7) = 42 - 26 \neq 0$$

we remark that $x = 7$ is a root of order of multiplicity $m = 2$

Then we will use the first modified Newton's Method

$$x_{n+1} = x_n - 2\,\frac{f(x_n)}{f'(x_n)} = x_n - 2\,\frac{x_n^3 - 13x_n^2 + 35x_n + 49}{3x_n^2 - 26x_n + 35}$$

$x_0 = 9$

$$x_1 = 9 - 2\,\frac{40}{44} = \frac{79}{11} \simeq 7.181818$$

$$x_2 = 9 - 2\,.\,\frac{0.270500}{3.0082} \simeq 8.8201$$

Q4) $f(x) = \ln x - x + 1.5$

$f(2) = 0.1931$

$f(3) = -0.4013$

$f(2) \cdot f(3) < 0$ then $\alpha \in (2,3)$

$x_1 = \frac{2+3}{2} = \frac{5}{2} \rightarrow f(5/2) = -0.083 < 0$

Then $\alpha \in [2, 2.5]$

$x_2 = \frac{2+2.5}{2} = 2.25 \rightarrow f(2.25) = 0.0609 > 0$

Then $\alpha \in [2.25, 2.5]$

$x_3 = \frac{2.25+2.5}{2} = 2.375$

Error Bound : $|\alpha - x_3| \leq \frac{b-a}{2^n} = \frac{3-2}{2^3} = \frac{1}{8}$

Q5) $\alpha = N^{1/3}$

$g(x) = ax + \frac{bN}{x^2}$

$g'(x) = a - \frac{2bN}{x^3}$ , $g''(x) = \frac{6bN}{x^4}$

$g(N^{1/3}) = N^{1/3} \rightarrow aN^{1/3} + \frac{bN}{N^{2/3}} = N^{1/3} \rightarrow a + b = 1$

$g'(N^{1/3}) = 0 \rightarrow a - 2b\frac{N}{N} = 0 \rightarrow a - 2b = 0$

$3b = 1 \rightarrow b = 1/3$

$\rightarrow a = 1 - b = 2/3$

$$g''(N^{1/3}) = \frac{2N}{N^{4/3}} \neq 0$$

$\rightarrow$ the convergence is quadratic