TIME: 3 Hours
M - 107

**DEPARTMENT OF MATHEMATICS**
**Final Examination (First Semester 1439-1440)**

**FULL MARKS: 80**

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**Question:1: (a) Solve the linear system by using Cramer's Rule**

**[6+6+6]**

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 3z &= -2 \\ x + 8z &= 8 \end{aligned}$$

**(b) Use row reduction to show that**

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$

**(c) Use Gauss- Jorden method to find the intersection point of three planes of equations**
$x+y+z=6,\ x-2y+3z=2$ and $x+2y-z=6$.

**Question:2( a) Find equation of the plane that is orthogonal to the plane** $3x+2y-z=4$
**[6+6+6]** **and contains points** $A(1,2,3)$ and $B(-1,3,1)$.

**( b) Given the points** $P(1,1,0)$, $Q(2,2,1)$ and $R(1,-1,2)$,

i. **Find the area of triangle PQR.**

ii. **Find the shortest distance from R to the line through P and Q.**

**(c) A particle moves on the curve** $x=2t^2,\ y=t^2-4t,\ z=3t-5$, **where t is time. Find components of velocity and acceleration in the direction of vector** $b=i-3j+2k$ **at time t = 1.**

**Question:3. (a) Show that** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ **does not exist.**

**[6+8+4]** **(b) The position vector of a moving particle at time t is** $r(t) = \langle \cos 2t,\ t^2,\ \sin 2t \rangle$.
**Find the tangential and normal components of acceleration at** $t=\dfrac{\pi}{4}$.

**(c) Find equations of the tangent plane and the normal line to the surface**
$z=x^2+y^3$ at point $(1,1,2)$.

**Question:4. (a) Let** $f(u,v)=u^2+3v^2,\ u=r\cos\theta,\ v=r\sin\theta$. **Use chain rule to find**
**[7+7]** $\dfrac{\partial f}{\partial \theta}$ and $\dfrac{\partial f}{\partial r}$ **also show that** $\dfrac{\partial^2 f}{\partial\theta\,\partial r}=\dfrac{\partial^2 f}{\partial r\,\partial\theta}$.

**(b)Find the directional derivative of** $f(x,y,z)=e^{xy+z}$ **at the point P** $(1,-1,1)$ **in** direction of $a=\langle 1,1,-2\rangle$. **Find the direction that produces maximum rate to change of** $f(x,y,z)$ **at P. What is the maximum rate of change at P?**

**Question:5. (a) Let the equation** $\ln z = xyz - x - y$ **define the implicit differentiable function**
**[6+6]** **z = g (x ,y). Find** $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

**(b) Find local extrema and saddle points if exist, of the function**
$f(x,y)=ye^x-3x-y+2$.

# Solution of M - 107 Final Examination (First Semester 1439-1440)

**Question:1: (a) Solve the linear system by using Cramer's Rule**

**[6+6+6]**

$$x + 2y + 3z = 1$$
$$2x + 5y + 3z = -2$$
$$x + 8z = 8$$

**(b) Use row reduction to show that**

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$

**(c) Use Gauss- Jordgn method to find the intersection point of three planes** of equations

$$x+y+z=6, \quad x-2y+3z=2 \text{ and } x+2y-z=6.$$

Solution. (a)

1 $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 8 \end{bmatrix}, \qquad AX = B$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix},$$ 1 $$\det A = 1(40-0) - 2(16-3) + 3(0-5) = 40 - 26 - 15 = 40 - 41 = -1$$

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 3 \\ 8 & 0 & 8 \end{bmatrix},$$ 1 $$\det A_1 = 0, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$\det A_2 = 1$$

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & 0 & 8 \end{bmatrix},$$ 1 $$\det A_3 = -1$$

2
$$x = \frac{\det A_1}{\det A} = 0$$
$$y = \frac{\det A_2}{\det A} = -1$$
$$z = \frac{\det A_3}{\det A} = 1$$

$x = 0, \; y = -1, \; z = 1$

(b)

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

6

$$= \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{vmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{vmatrix} R_4 - R_3$$

$$= a(b-a)(c-b)(d-C)$$

(c)

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -2 & 3 & 2 \end{array}\right] \equiv \cdots\cdots \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$ 5

**Question:2( a)** **Find equation of the plane that is orthogonal to the plane** $3x+2y-z=4$
**[6+6+6]** **and contains points** $A(1,2,3)$ and $B(-1,3,1)$.

**( b) Given the** points $P(1,1,0)$, $Q(2,2,1)$ and $R(1,-1,2)$,

i. **Find the area of triangle PQR.**

ii. **Find the shortest distance from R to the line through P and Q.**

**(c) A particle moves on the curve** $x=2t^2$, $y=t^2-4t$, $z=3t-5$, **where t is time. Find components of velocity and acceleration in the direction of vector** $b=i-3j+2k$ **at time t = 1.**

Solution

(1) (a) vector in the required plane $\overrightarrow{AB} = \langle -2, 1, -2 \rangle$

(1) Vector normal to given plane $n = \langle 3, 2, -1 \rangle$

vector normal to required plane

(2) $$N = \overrightarrow{AB} \times n = \begin{vmatrix} i & j & k \\ -2 & 1 & -2 \\ 3 & 2 & -1 \end{vmatrix} = \langle 3, -8, -7 \rangle$$

(2) Equation the plane containing point $A(1,2,3)$ and normal vector $N$ is $3(x-1) - 8(y-2) - 7(z-3) = 0$

$$3x - 8y - 7z + 34 = 0$$

(b) $\overrightarrow{PQ} = \langle 1, 1, 1 \rangle$, $\overrightarrow{PR} = \langle 0, -2, 2 \rangle$

(1) Area of $\triangle PQR = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

(2) $$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = \langle 4, -2, -2 \rangle$$

$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{16+4+4} = \sqrt{24}$

(ii) Shortest distance from R is (2)

$$d = \frac{\|PQ \times PR\|}{\|PR\|} = \frac{\sqrt{24}}{\sqrt{3}} = \sqrt{8}$$

(1) (i) Area of $\triangle PQR = \frac{1}{2}\sqrt{24} = \frac{2}{2}\sqrt{6} = \sqrt{6}$ unit$^2$

(c) $r(t) = \langle 2t^2, t^2-4t, 3t-5 \rangle$

(1) • velocity $v(t) = r'(t) = \langle 4t, 2t-4, 3 \rangle$

(1) • accelaration $a(t) = r''(t) = \langle 4, 2, 0 \rangle$

$b = i - 3j + 2k$. $\|b\| = \sqrt{1+9+4} = \sqrt{14}$

(1) • $$\text{Comp}_b\, v(t) = \frac{v(t)\cdot b}{\|b\|} = \frac{-2t+18}{\sqrt{14}}$$

(1) • $$\text{Comp}_b\, a(t) = \frac{a(t)\cdot b}{\|b\|} = \frac{-2}{\sqrt{14}}$$

• At $t = 1$. (1) $$\text{Comp}_b\, v(1) = \frac{16}{\sqrt{14}}$$

(1) $$\text{Comp}_b\, a(1) = \frac{-2}{\sqrt{14}}$$

**Question:3. (a)** Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ does not exist.

**[6+8+6]** **(b)** The position vector of a moving particle at time t is $r(t) = <\cos 2t,\ t^2,\ \sin 2t>$.

Find the tangential and normal components of acceleration at $t = \frac{\pi}{4}$.

**(c)** Find equations of the tangent plane and the normal line to the surface $z = x^2 + y^3$ at point $(1,1,2)$.

Solution (a)

Limit along Path $y=0$

$\lim_{(x,0)\to(0,0)} \frac{0}{x^4+0} = 0$

(2) Limit along Path $x=0$

$\lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$

(2) Limit along Path $y = x^2$

$\lim_{(x,x^2)\to(0,0)} \frac{x^2 \cdot x^2}{x^4+x^4} = \lim_{(x,x^2)\to(0,0)} \frac{1}{2} = \frac{1}{2}$

(2) • Hence $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ does not exist.

(b) $r(t) = <\cos 2t,\ t^2,\ \sin 2t>$

(1) $r'(t) = <-2\sin 2t,\ 2t,\ 2\cos 2t>$

(1) $r''(t) = <-4\cos 2t,\ 2,\ -4\sin 2t>$

(1) $\|r'(t)\| = 2\sqrt{1+t^2}$,

$r'(t) \cdot r''(t) = 4t$

(2) $\|r'(t) \times r''(t)\|^2 = 64t^2 + 80$

(1) • $a_T = \frac{r' \cdot r''}{\|r'\|} = \frac{4t}{2\sqrt{1+t^2}} = \frac{2t}{\sqrt{1+t^2}}$

(1) • $a_N = \frac{\|r' \times r''\|}{\|r'\|} = \frac{2\sqrt{4t^2+5}}{\sqrt{1+t^2}} = 2\sqrt{\frac{4t^2+5}{1+t^2}}$

• At $t = \frac{\pi}{4}$ (1) $a_T = \frac{2\pi}{\sqrt{16+\pi^2}}$

$a_N = 2\sqrt{\frac{4\pi^2+80}{\pi^2+16}}$

(c) $F(x,y,z) = x^2 + y^3 - z = 0$

(2) $\nabla F = <2x,\ 3y^2,\ -1>$

$N = \nabla F(1,1,2) = <2,\ 3,\ -1>$

• Equation of tangent plane (1)

$2(x-1) + 3(y-1) - 1(z-2) = 0$

$2x + 3y - z = 3$

• Equation of normal line

(1) $x = 1 + 2t$

$y = 1 + 3t$

$z = 2 - t, \quad t \in R.$

**Question:4. (a)** Let $f(u,v)=u^2+3v^2,\ u=r\cos\theta,\ v=r\sin\theta$. **Use chain rule to find**

**[7+7]** $\frac{\partial f}{\partial\theta}$ **and** $\frac{\partial f}{\partial r}$ **also show that** $\frac{\partial^2 f}{\partial\theta\,\partial r}=\frac{\partial^2 f}{\partial r\,\partial\theta}$.

**(b)Find the directional derivative of** $f(x,y,z)=x^2+y^3-z$ **at the point P (1, 1, 2) in direction of** $a=<1,0,-2>$. **Find the direction that produces maximum rate to change of** $f(x,y,z)$ **at P. What is the maximum rate of change at P?**

Solution (a)

$$\frac{\partial f}{\partial\theta}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial\theta}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial\theta}$$

$$=(2u)(-r\sin\theta)+(6v)(r\cos\theta)$$

$$=(2r\cos\theta)(-r\sin\theta)+(6r\sin\theta)(r\cos\theta)$$

$$=-2r^2\cos\theta\sin\theta+6r^2\sin\theta\cos\theta$$

(2) $=4r^2\cos\theta\sin\theta$ , (1) $\frac{\partial^2 f}{\partial r\,\partial\theta}=8r\cos\theta\sin\theta \longrightarrow 1$

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial r}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial r}$$

$$=(2u)(\cos\theta)+(6v)(\sin\theta)$$

$$=(2r\cos\theta)\cos\theta+(6r\sin\theta)(\sin\theta)$$

(2) $=2r\cos^2\theta+6r\sin^2\theta$

(1) $\frac{\partial^2 f}{\partial\theta\,\partial r}=-4r\cos\theta\cdot\sin\theta+12r\sin\theta\cos\theta=8r\sin\theta\cdot\cos\theta\rightarrow 2$

(1) From Eq. 1 and Eq. 2 $\frac{\partial^2 f}{\partial r\cdot\partial\theta}=\frac{\partial^2 f}{\partial\theta\,\partial r}$

(b). $f(x,y,z)=e^{xy+z}$

(2) $\nabla f(x,y,z)=\langle ye^{xy+z},\ xe^{xy+z},\ e^{xy+z}\rangle$

(1) $\nabla f(1,-1,1)=\langle -1,1,1\rangle$

$a=\langle 1,1,-2\rangle,\ \|a\|=\sqrt6$

$u=\frac{a}{\|a\|}=\langle \frac{1}{\sqrt6},\frac{1}{\sqrt6},\frac{-2}{\sqrt6}\rangle$

(2) $D_u f(1,-1,1)=-\frac{1}{\sqrt6}+\frac{1}{\sqrt6}-\frac{2}{\sqrt6}=-\frac{2}{\sqrt6}$

(1) Maximum rate of change at $P(1,-1,1)$ occurs in direction of $\nabla f(1,-1,1)=\langle -1,1,1\rangle$

(1) Value of maximum rate of change at $P(1,-1,1)$ is $\|\nabla f(1,-1,1)\|=\sqrt3$

**Question:5.** **(a) Let the equation** $\ln z = xyz - x - y$ **define the implicit differentiable function**
**[6+6]** **z =** $g(x,y)$**. Find** $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

**(b) Find local extrema and saddle points if exist, of the function** $f(x,y) = ye^x - 3x - y + 2$.

Solution.

(a) $F(x,y,z) = xyz - x - y - \ln z = 0$

2 $\begin{cases} F_x = yz - 1 \\ F_y = xz - 1 \\ F_z = xy - \frac{1}{z} = \frac{xyz - 1}{z} \end{cases}$

2 • $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz - 1}{\frac{xyz-1}{z}} = \frac{z - yz^2}{xyz - 1}$

2 • $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz - 1}{\frac{xyz-1}{z}} = \frac{z - xz^2}{xyz - 1}$

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(b). 2 $\begin{cases} f_x = ye^x - 3 \\ f_y = e^x - 1 \end{cases}$

• critical points $f_x = 0$, $f_y = 0$

$ye^x - 3 = 0$

$e^x - 1 = 0$

$e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow e^x = e^0 \Rightarrow x = 0, \quad y = 3$

1 • $(0, 3)$

$f_{xx} = ye^x, \quad f_{xy} = e^x, \quad f_{yy} = 0$

1 • $D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 0 - e^{2x}$

1 • $D(0,3) = -1 < 0 \Rightarrow$ There is saddle point

$f(0,3) = 3 - 0 - 3 + 2 = 2$

1 • saddle point is $(0, 3, 2)$.