ASYMPTOTICS OF SOLUTIONS OF NONLINEAR ABEL-VOLTERRA q-INTEGRAL EQUATIONS NEAR ZERO

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ABSTRACT. lalalal

1. Introduction

In [?], Mansour proved the existence and uniqueness of positive continuous solutions of the nonlinear Fredholm q-integral equations

$$\phi(x) = \lambda(x)\int_0^1 (qt/x;q)_{\alpha-1}\phi^p(t)\,d_qt \quad (0 \le x \le 1) \tag{1.1}$$

and

$$\phi(x) = f(x) + \lambda(x)\int_0^1 (qt/x;q)_{\alpha-1}\phi^p(t)\,d_qt \quad (0 \le x \le 1) \tag{1.2}$$

where both of λ and f are positive continuous functions on $[0,1]$ and $0 < |p| < 1$. S Replace p, and ϕ by $\frac{1}{m}$, and ϕ^m on (1.1) and (1.2), respectively, where $m \notin \{0,-1,-2,\ldots\}$. This yields the Fredholm q-integral equations

$$\phi^m(x) = \lambda(x)\int_0^1 (qt/x;q)_{\alpha-1}\phi(t)\,d_qt \quad (0 \le x \le 1) \tag{1.3}$$

and

$$\phi^m(x) = f(x) + \lambda(x)\int_0^1 (qt/x;q)_{\alpha-1}\phi(t)\,d_qt \quad (0 \le x \le 1) \tag{1.4}$$

In this paper, we investigate the asymptotics of solutions of (1.3) and (??) when f and λ have following power asymptotic near zero

$$\lambda(x) \sim x^{\alpha pm}\sum_{k=-l}^{\infty}\lambda_k x^{\alpha k}, \tag{1.5}$$

with $\lambda_{-l} \neq 0$ and

$$f(x) \sim x^{\alpha pm}\sum_{k=-n}^{\infty} f_k x^{\alpha k} \tag{1.6}$$

2. Preliminaries and q-notations

Let q be a positive number which is less than 1, $\mathbb{N}$ be the set of all positive integers, and $\mathbb{N}_0$ be the set of all nonnegative integers. In the following, we follow the notations and notions of q-hypergeometric functions, the q-gamma function $\Gamma_q(x)$, Jackson q-exponential functions $e_q(x)$, and the q-shifted factorial as in [3, 4]. By a q-geometric set A we mean a set that satisfies if $x \in A$ then $qx \in A$. Let f be a function defined on a q-geometric set A. The $q-$difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{2.1}$$

Jackson [5] introduced an integral denoted by

$$\int_a^b f(x)\, d_qx$$

as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t)\, d_qt := \int_0^b f(t)\, d_qt - \int_0^a f(t)\, d_qt,\ a, b \in \mathbb{C}, \tag{2.2}$$

where

$$\int_0^x f(t)\, d_qt := (1-q) \sum_{n=0}^{\infty} xq^n f(xq^n),\ x \in \mathbb{C}, \tag{2.3}$$

provided that the series at the right-hand side of (2.3) converges at $x = a$ and b. A q-analogue of the Riemann-Liouville fractional integral operator is introduced in [1] by Al-Salam through

$$I_q^{\alpha} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x; q\big)_{\alpha-1} f(t)\, d_qt, \tag{2.4}$$

$\alpha \notin \{-1, -2, \ldots\}$. Using (2.3), we obtain

$$I_q^{\alpha} f(x) := x^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{(q^{\alpha}; q)_k}{(q; q)_k} f(xq^k), \tag{2.5}$$

which is valid for all α. Let $f : A \to \mathbb{R}$ be a function. We say that f is q-decreasing on A if

$$f(x) \leq f(qx) \quad \text{for all } x \in A.$$

If

$$f(x) \geq f(qx) \quad \text{for all } x \in A$$

then we call f q-increasing on A. If

$$D_q f(x) \geq 0 \text{ for all } x \in A \cap \mathbb{R}^+ \text{ and } D_q f(x) \leq 0 \text{ for all } x \in A \cap \mathbb{R}^-,$$

then f is q-increasing on A. Similarly, if

$$D_q f(x) \leq 0 \text{ for all } x \in A \cap \mathbb{R}^+ \text{ and } D_q f(x) \geq 0 \text{ for all } x \in A \cap \mathbb{R}^-,$$

then f is q-decreasing on A. Let $B[0, a]$ be the space of all bounded functions defined on $[0, a]$ and $B^+[0, a]$ be the space of all functions $f \in B[0, 1]$ such that

$$\inf \{f(x) : x \in [0, a]\} > 0.$$

The q-translation operator is introduced by Ismail in [4] and is defined on monomials by

$$\varepsilon^y x^n := x^n(-y/x; q)_n, \tag{2.6}$$

and it is extended to polynomials as a linear operator.

3. Asymptotic Solutions Near Zero

The following theorem is proved in [7]. Therefore we introduce it without a proof.

Theorem 3.1. *Let $p \in \mathbb{Z}, \alpha \in \mathbb{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If the function $\varphi(x)$ has the asymptotic relation*

$$\phi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0), \tag{3.1}$$

then for $m \in \mathbb{R}$ with $m \neq 0, -1, -2, ...$, there holds the asymptotic

$$\phi^m(x) \sim x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{p,k} x^{\alpha k} \ (x \to 0) \tag{3.2}$$

where the coefficients $\Phi_{p,k}$ are expressed in terms of the coefficients φ_k:

$$\Phi_{p,0} = \binom{m}{0} \varphi_p^m,$$

$$\Phi_{p,1} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+1},$$

$$\Phi_{p,2} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+2} + \binom{m}{2} \varphi_p^{m-2} \varphi_{p+1}^2,$$

(3.3)

$$\Phi_{p,3} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+3} + \binom{m}{2}\binom{2}{1} \varphi_p^{m-2} \varphi_{p+1}\varphi_{p+2} + \binom{m}{3} \varphi_p^{m-3} \varphi_{p+1}^3,$$

$$\Phi_{p,4} = \binom{m}{1}\varphi_p^{m-1}\varphi_{p+4} + \binom{m}{2}\varphi_p^{m-2}\left[\varphi_{p+2}^2 + \binom{2}{1}\varphi_{p+1}\varphi_{p+3}\right] + \binom{m}{3}\binom{3}{1}\varphi_p^{m-3}\varphi_{p+1}^2\varphi_{p+2} + \binom{m}{4}\varphi_p^{m-4}\varphi_{p+1}^4,$$

etc.

In [7], Kilbas and Saigo proved that if in Theorem 3.1 $m \in \{2, 3, \ldots\}$, then

$$\Phi_{p,k} = \sum_{i_0=0}^{m-1}\sum_{i_1,i_2,\ldots,i_j}\frac{m!}{i_0!i_1!i_2!\ldots i_j!}\phi_p^{i_0}\phi_{p+1}^{i_1}\ldots\phi_{p+j}^{i_j}$$

where the summation is taken over all non-negative integers $i_1,\ i_2, \ldots, i_j$ such that

$$0 \le i_1 \le i_2 \le \ldots \le i_j \le k,$$

$$i_0 + i_1 + \ldots + i_j = m,\ i_1 + 2i_2 + \ldots + ji_j = k.$$

Theorem 3.2. *Let $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ be such that $\alpha p > -1$. If*

$$\phi(x) \sim \sum_{j=p}^{\infty}\varphi_j x^{\alpha j} \quad (x \to 0), \tag{3.4}$$

then

$$I_q^{\alpha}\phi(x) \sim x^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\frac{\Gamma_q(j\alpha+1)}{\Gamma_q(j\alpha+\alpha+1)}\ (x \to 0).$$

Proof. First we consider (1.4), where $\lambda(x)$ and $f(x)$ have the asymptotics (1.5) and (1.6), respectively. We will seek an asymptotic solution $\varphi(x)$ of (1.4) in the form (3.4). From (2.5)

$$I_q^{\alpha}\phi(x) = x^{\alpha}(1-q)^{\alpha}\sum_{k=0}^{\infty}q^k\frac{(q^{\alpha};q)_k}{(q;q)_k}\phi(xq^k).$$

Hence, applying (3.1) we have

$$\begin{aligned} I_q^{\alpha}\phi(x) &\sim x^{\alpha}(1-q)^{\alpha}\sum_{k=0}^{\infty}q^k\frac{(q^{\alpha};q)_k}{(q;q)_k}\sum_{j=p}^{\infty}\varphi_j x^{\alpha j}q^{k\alpha j} \\ &= x^{\alpha}(1-q)^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\sum_{k=0}^{\infty}q^{k(\alpha j+1)}\frac{(q^{\alpha};q)_k}{(q;q)_k} \\ &= x^{\alpha}(1-q)^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\frac{(q^{\alpha j+\alpha+1})}{(q^{\alpha}j+1)}, \end{aligned}$$

where we applied the q-binomial theorem in the last step, cf. [3, xvii]. Consequently,

$$I_q^\alpha \phi(x) \sim x^\alpha \sum_{j=p}^{\infty} \varphi_j x^{j\alpha} \frac{\Gamma_q(j\alpha+1)}{\Gamma_q(j\alpha+\alpha+1)},$$

and the theorem follows.

□

In view of the asymptotic (1.5) and the general properties of asymptotic expansions, see [8, Chapter 1], we have

$$\frac{\lambda(x)x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\phi(t)\, d_q t$$

$$\sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \right) x^{\alpha k} \ (x \to 0).$$

Then, taking into account Theorem 3.1 and Lemma 1.6, we obtain (3.5)

$$x^{\alpha(l-n-1)m} \sum_{k=0}^{\infty} \Phi_{l-n-1,k} x^{\alpha k} \sim$$

$$x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \ (x \to 0).$$

Theorem 3.3. *Let $\alpha > 0$, p, $m \in \mathbb{R}$ $(m \neq 0, -1, -2, ...)$ be such that $\alpha p > -1$. Assume that as well as $l, n, r := (l-n-p-1)m \in \mathbb{Z}$ such that $l-n-1 > -1/\alpha$ and $r \geq -n$. Let $\lambda(x)$ and $f(x)$ have the asymptotics* (1.5) *and* (1.6) *and the coefficients φ_k satisfy*

$$\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k = 0 \tag{3.6}$$

$$(k = -n, -n+1, ..., r-1),$$

and

$$\Phi_{l-n-1,k-r} = \sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k \tag{3.7}$$

$$(k = r, r+1, ...).$$

Then (1.4) *is asymptotically solvable in $\mathcal{L}_q^1(0,a)$ for some $a > 0$ and its asymptotic solution near zero has the form* (3.1).

Proof. Suppose $r = (l-n-p-1)m \in \mathbb{Z}$ for $m \in \mathbb{R}, (m \neq 0, -1, -2, ...)$ such that $r \geq -n$. Then (3.5) is equivalent to

(3.8)
$$x^{\alpha(r+pm)} \sum_{k=0}^{\infty} \Phi_{l-n-1,k} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k}.$$

Make the substation $\mu = k + r$ on the left hand side of the last relation and then replace μ by k, this gives

(3.9)
$$x^{\alpha pm} \sum_{k=r}^{\infty} \Phi_{l-n-1,k-r} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k}.$$

Hence, it follows from (3.9) that the coefficients φ_k satisfy (3.6) and (3.7). Then (1.4) is asymptotically solvable and its asymptotic solution near zero is given by (3.1). □

Theorem 3.4. *Let $\alpha > 0, m > 0$ and let $p, l, n \in \mathbb{Z}$ be such that $n = l-p-1 < 0$ and $(p-l+1)/m \in \mathbb{Z}$ and set $r = p+(p-l+1)/m > -1/\alpha$. Let $\lambda(x)$ and $f(x)$ have the asymptotics (1.5) and (1.6), respectively. Moreover, let the coefficients φ_k satisfy the relations*

$$\Phi_{r,k+l-p-1} = f_k \tag{3.10}$$
$$(k = p-l+1, p-l+2, ..., r-l),$$

and

$$\Phi_{r,k+l-p-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1} \varphi_i + f_k \tag{3.11}$$
$$(k = r-l+1, r-l+2, ...).$$

Then (1.4) is asymptotically solvable in $\mathcal{L}_q^1(0, a)$ and its asymptotic solution near zero has the form

$$\phi(x) \sim \sum_{k=r}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0). \tag{3.12}$$

Proof. We assume

$$\phi(x) \sim \sum_{k=r}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0),$$

and we shall prove that the coefficients φ_k are given by (3.10), (3.11). Applying the same arguments as in Theorem 3.3, we came to the asymptotic relation

(3.13)

$$x^{\alpha r m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} \sim x^{\alpha p m} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} + x^{\alpha p m} \sum_{k=p-l-1}^{\infty} f_k x^{\alpha k}.$$

Since $m(r-p) \in \mathbb{N}$, the left hand side of (3.13) can be written as

$$\begin{aligned} x^{\alpha r m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} &= x^{\alpha p m + \alpha(r-p)m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} \\ &= x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha(k+(r-p)m)}. \end{aligned}$$

Make the substation $\mu = k + (r-p)m$ on the last series and then replace μ by k, this gives

(3.14)

$$x^{\alpha p m} \sum_{k=(r-p)m}^{\infty} \Phi_{r,k-(r-p)m} x^{\alpha k} \sim x^{\alpha p m} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} + x^{\alpha p m} \sum_{k=p-l+1}^{\infty} f_k x^{\alpha k}.$$

Equating the coefficients of $x^{\alpha k}$ on (3.14) give (3.10) and (3.11). Then (1.4) is asymptotically solvable in $\mathcal{L}_q^1(0,a)$ for some $a > 0$ and its asymptotic solution is in the form (3.12). □

Corollary 3.5. *Let $\alpha > 0$, $m > 0$ and let l be a positive integer such that $l/m \in \mathbb{Z}$ and set $r = -1 + l/m$. Let*

$$\lambda(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} \lambda_k x^{\alpha k}, \ f(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} f_k x^{\alpha k} \ (x \to 0),$$

with $\lambda_l \neq 0$, $f_l \neq 0$. Then (1.4) *is asymptotically solvable and its asymptotic solution near zero has form* (3.12).

Proof. The proof follows from Theorem 3.4 by substituting with $p = -1$, $n = l$ and $f_k = 0$ $(k = -l, -l+1, ..., -l+r)$ in equations (1.5) and (1.6). Hence, the coefficients φ_k satisfy

$$\Phi_{r,k+l} = f_k \quad (k = -l, -l+1, ..., -l+r), \tag{3.15}$$

and

$$\Phi_{r,k+l} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k \quad (k = r-l+1, r-l+2, ...). \tag{3.16}$$

Then (1.4) is asymptotically solvable in $\mathcal{L}_q^1(0,a)$ for some $a > 0$ and its asymptotic solution near zero has the form (3.12). □

Theorem 3.6. *Let $\alpha > 0, m < 1\,(m \neq 0, -1, -2, ...)$ and let $p, l, n \in \mathbb{Z}$ be such that $n = l - p - 1 < 0$ and $(p-l+1)/(1-m) \in \mathbb{Z}$ and let $r = p-(p-l+1)/(1-m) > -1/\alpha$. Let $\lambda(x)$and $f(x)$ have asymptotics* (1.5) *and* (1.6) *and the coefficients φ_k satisfy*

$$\Phi_{r,k+l-r-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \quad (k = r-l+1, r-l+2, ..., p-l), \tag{3.17}$$

and

$$\Phi_{r,k+l-r-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k \quad (k = p-l+1, p-l+2, ...). \tag{3.18}$$

Then (1.4) *is asymptotically solvable and its asymptotic solution near zero has the form* (3.12).

Proof. Applying Theorem (3.4) we have

$$x^{\alpha pm} \sum_{k=r-l+1}^{\infty} \Phi_{r,k-(r-l+1)} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=p-l+1}^{\infty} f_k x^{\alpha k}.$$

Then the coefficients φ_k satisfy (3.17) and (3.18). Then (1.4) is asymptotically solvable and its asymptotic solution near zero has the form (3.12) □

4. Asymptotic of the Solution in Some Special Cases

In this section we give asymptotic solutions of (1.4) when $\lambda(x)$ and $f(x)$ have the special case:

$$\lambda(x) = \lambda x^{\alpha(pm-l)} \text{ and } f(x) = -x^{\alpha pm} \sum_{k=-n}^{N} f_k x^{\alpha k}.$$

Hence (1.4) takes the form

(4.1)

$$\phi^m(x) = \frac{\lambda x^{\alpha(pm-l)}}{\Gamma_q(\alpha)} x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt \, - x^{\alpha pm} \sum_{k=-n}^{N} f_k x^{\alpha k},$$

where $0 < x < a \leq \infty$, $\lambda \neq 0$ and $N \geq -n$.

Theorem 4.1. *Let $\alpha > 0, p, m \in \mathbb{R}\,(m \neq 0, -1, -2, ...)$ and let l, n, $(l-n-p-1)m \in \mathbb{Z}$ be such that $l-n-1 > -1/\alpha$ and $(l-n-p-1)m \geq -n$.*

(1) *When $r = (l-n-p-1)m > N$ and the coefficients φ_k satisfy*

$$\varphi_k = 0 \; (k = N+l, N+l+1, ..., r+l-2), \tag{4.2}$$

$$\varphi_k = \frac{-f_{k-l+1}\Gamma_q(\alpha k+\alpha+1)}{\lambda \Gamma_q(\alpha k+1)} \tag{4.3}$$

$$(k = -n+l-1, -n+l, ..., N+l-1),$$

and

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda \Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{4.4}$$

$$(k = r+l-1, r+l, ...).$$

Then (4.1) is asymptotically solvable in $\mathcal{L}_q^1(0,a)$ for some $a > 0$ and its asymptotic solution near zero has the form

$$\phi(x) \sim \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1} x^{\alpha k} + \sum_{k=r+l-1}^{\infty} \varphi_k x^{\alpha k}. \tag{4.5}$$

(2) *When $-n < r \leq N$ and the coefficients φ_k satisfy*

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k - f_{k-l+1} \tag{4.6}$$

$$(k = r+l-1, r+l, ..., N+l-1),$$

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{4.7}$$

$$(k = N+l, N+l+1, ...).$$

Then (4.1) *is asymptotically solvable in* $\mathcal{L}_q^1(0,a)$ *for some* $a > 0$ *and its asymptotic solution near zero has the form*

$$\phi(x) \sim \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1}x^{\alpha k} + \sum_{k=r+l-1}^{\infty} \varphi_k x^{\alpha k}. \tag{4.8}$$

(3) *When* $r = -n$ *and the coefficients* φ_k *satisfy*

$$\Phi_{l-n-1,k-l+1+n} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k - f_{k-l+1} \tag{4.9}$$

$$(k = -n+l-1, -n+l, ..., N+l-1),$$

$$\Phi_{l-n-1,k-l+1+n} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{4.10}$$

$$(k = N+l, N+l+1, ...).$$

Then (4.1) *is asymptotically solvable in* $\mathcal{L}_q^1(0,a)$ *and its asymptotic solution near zero has the form* (3.1).

Proof. From (1.5) and (1.6) we obtain $\lambda_{-l} = \lambda, \lambda_j = 0$ for $j > l$, and $f_k = 0$ for $k > N$. Hence, conditions (4.2)-(4.4) imply that the conditions (3.6) and (3.7) of Theorem 3.3 are satisfied. Hence ϕ has the asymptotic (3.1) where the coefficients are given by (4.2)- (4.4). That is ϕ has the asymptotic (4.5). This proves (1) of the Theorem. The proofs of the points (2), (3) are similar to the proof of (1) and so they are omitted. □

Corollary 4.2. *Under the assumptions* (4.2)- (4.4) *of Theorem 4.1(1), the solution* $\phi(x)$ *of* (4.1) *has the asymptotic*

$$\phi(x) = \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1}x^{\alpha k}$$

$$+\varphi_{r+l-1}x^{\alpha(r+l-1)} + O(x^{\alpha(r+l)})\ (x \to 0), \tag{4.11}$$

where

$$\varphi_{r+l-1} = \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \times \left(\frac{\Gamma_q(\alpha(l-n)+1)f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^m. \tag{4.12}$$

Furthermore, if $N > -n$, we have

$$\phi(x) = \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k + \alpha + 1)}{\lambda \Gamma_q(\alpha k + 1)} f_{k-l+1} x^{\alpha k}$$

$$+\varphi_{r+l-1} x^{\alpha(r+l-1)} + \varphi_{r+l} x^{\alpha(r+l)} + O(x^{\alpha(r+l+1)}) \ (x \to 0), \tag{4.13}$$

where φ_{r+l-1} is given by (4.12) and

$$\begin{aligned} \varphi_{r+l} = & \frac{m\Gamma_q(\alpha(r+l+1)+1)}{\lambda\Gamma_q(\alpha(r+l)+1)} \left(\frac{\Gamma_q(\alpha(l-n)+1) f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)} \right)^{m-1} \\ & \times \frac{\Gamma_q(\alpha(l-n+1)+1) f_{-n+1}}{\lambda\Gamma_q(\alpha(l-n)+1)}. \end{aligned} \tag{4.14}$$

Proof. Substitute with $k = r + l - 1$ in (4.4). This gives

$$\begin{aligned} \varphi_{r+l-1} &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \Phi_{l-n-1,0} \\ &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \varphi_{l-n-1}^{m}. \end{aligned}$$

Then put $k = -n$ in (3.6) we get

$$\varphi_{l-n-1} = \frac{\Gamma_q(\alpha(l-n)+1)}{\lambda\Gamma_q(\alpha(l-n-1)+1)} f_{-n}.$$

Then we are done. Similarly if $N > -n$. □

Corollary 4.3. *Under the assumptions of Theorem 4.1(2), the solution $\phi(x)$ of (4.1) has the asymptotic*

$$\phi(x) = \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k + \alpha + 1)}{\lambda \Gamma_q(\alpha k + 1)} f_{k-l+1} x^{\alpha k}$$

$$+\varphi_{r+l-1} x^{\alpha(r+l-1)} + O(x^{\alpha(r+l)}) \ (x \to 0), \tag{4.15}$$

where

$$\varphi_{r+l-1} = \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \times \left\{ f_r + \left(\frac{\Gamma_q(\alpha(l-n)+1) f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)} \right)^m \right\}. \tag{4.16}$$

Furthermore, if $N \geq r > -n + 1$, we have

$$\phi(x) = \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k + \alpha + 1)}{\lambda \Gamma_q(\alpha k + 1)} f_{k-l+1} x^{\alpha k}$$

$$+\varphi_{r+l-1}x^{\alpha(r+l-1)}+\varphi_{r+l}x^{\alpha(r+l)}+O(x^{\alpha(r+l+1)})\ (x\to 0), \tag{4.17}$$

where φ_{r+l-1} *is given by*(4.16) *and*

$$\begin{aligned}\varphi_{r+l} = & \frac{m\Gamma_q(\alpha(r+l+1)+1)}{\lambda\Gamma_q(\alpha(r+l)+1)}\times\left(\frac{\Gamma_q(\alpha(l-n)+1)f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^{m-1}\\ & \times\frac{\Gamma_q(\alpha(l-n+1)+1)f_{-n+1}}{\lambda\Gamma_q(\alpha(l-n)+1)}.\end{aligned} \tag{4.18}$$

Proof. Substitute with $k=r+l-1$ in (4.7). This gives

$$\begin{aligned}\varphi_{r+l-1} &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)-1)}\Phi_{l-n-1,0}\\ &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)}\varphi^m_{l-n-1}\\ &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)}\left\{f_r+\left(\frac{\Gamma_q(\alpha(l-n)+1)f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^m\right\}.\end{aligned}$$

Similarly if $N\geq r>-n+1$ □

Corollary 4.4. *Under the assumptions of Theorem 4.1(3), the solution* $\phi(x)$ *of*(4.1) *has the asymptotic*

$$\phi(x)=Ax^{\alpha(l-n-1)}+O(x^{\alpha(l-n)})\ (x\to 0), \tag{4.19}$$

where $\xi=A$ *is a solution of the equation*

$$\xi^m-\frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\xi+f_{-n}=0. \tag{4.20}$$

Furthermore, if $N\geq -n+1$ *and*

$$\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)}\neq mA^{m-1},$$

we have

$$\phi(x)=Ax^{\alpha(l-n-1)}+Bx^{\alpha(l-n)}+O(x^{\alpha(l-n+1)})\ (x\to 0), \tag{4.21}$$

where

$$B=\left[\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)}-mA^{m-1}\right]^{-1}f_{-n+1}.$$

Proof. Substitute with $k=l-n-1$ in (4.9). This gives

$$\Phi_{l-n-1,0}=\frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\varphi_{l-n-1}-f_{-n},$$

$$\varphi_{l-n-1}^{m} - \frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\varphi_{l-n-1} + f_{-n} = 0.$$

Then $\varphi_{l-n-1} = A$ is a solution of equation (4.20). Now we prove (4.21) if $N \geq -n+1$ and

$$\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)} \neq mA^{m-1},$$

put $k = l-n$ in (4.9). We obtain

$$\Phi_{l-n-1,1} = \frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n+1)+1)}\varphi_{l-n} - f_{-n+1},$$

$$m\varphi_{l-n-1}^{m-1}\varphi_{l-n} = \frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n+1)+1)}\varphi_{l-n} - f_{-n+1},$$

where we used (3.3). Since $A = \varphi_{l-n-1}$ and $B = \varphi_{l-n}$, then we have

$$B = \left[\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)} - mA^{m-1}\right]^{-1} f_{-n+1}.$$

□

In the remaining of this section we derive an exact solution of the equation

$$\phi^m(x) = \frac{\lambda x^{\alpha(pm-l)}}{\Gamma_q(\alpha)} x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt \,-bx^{\alpha(pm-n)} \tag{4.22}$$

$$(0 < x < a \leq \infty).$$

From Corollary 4.4, the solution $\phi(x)$ of (4.22) has the asymptotic (4.19) near zero, where $\xi = A$ is a solution of equation

$$\xi^m - \frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\xi + b = 0.$$

Theorem 4.5. *Let $\alpha > 0$, $\beta > -1$ ($\beta \neq 0$), and $l \in \mathbb{R}$ with $l \neq -\alpha$ and $l \neq -\alpha-\beta$. For $a, b \in \mathbb{R}$ ($a \neq 0$) let the equation*

$$\xi^{1+(l+\alpha)/\beta} - \frac{\lambda\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}\xi - b = 0, \tag{4.23}$$

be solvable and let $\xi = c$ be its solution. Then the nonlinear integral equation

$$\phi^{1+(l+\alpha)/\beta}(x) = \frac{\lambda x^l}{\Gamma_q(\alpha)} x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt + bx^{\alpha+\beta+l} \tag{4.24}$$

$$(0 < x < a \leq \infty)$$

is solvable and its solution is given by

$$\phi(x) = cx^{\beta}. \tag{4.25}$$

Proof. We apply Corollary 4.4 with $m = 1 + \frac{\alpha+\gamma}{\beta}$, $\gamma = \alpha(pm - l)$, and $\alpha(pm - n) = \alpha + \beta + \gamma$. Then the solution is given by $\phi(x) = cx^{\beta} + O(x^{\beta+\alpha})$. But a direct substitution verifies that $\phi(x) = cx^{\beta}$ is a solution of

$$\xi^m - \frac{\lambda\Gamma_q(l-n-\alpha+1)}{\Gamma_q(n-l+1)}\xi - b = 0.$$

□

Now we consider the homogeneous equation associated with (4.24) which is

$$\phi^{1+(l+\alpha)/\beta}(x) = \frac{\lambda x^l}{\Gamma_q(\alpha)} x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt \;(0 < x < a \le \infty). \tag{4.26}$$

According to Theorems 3.1 and (**??**) we obtain the following result

Theorem 4.6. *Let the conditions of Theorem 4.5 are satisfied and let* $\xi = c$ *be the unique solution of* (4.23).

(i) *If* $-1 < (l+\alpha)/\beta < 0$*, then* (4.25) *is the unique solution of* (4.26) *in the space* $C[0,a]$ *for some* $a > 0$*. If in additionally,* $\lambda, b,$ *and* C *are positive numbers, then this solution belongs to* $C^+[0,d]$*.*

(ii) *If* $(\alpha+l)/\beta > 0$*,* $a, b,$ *and* c *are positive numbers, then* (4.25) *is the unique solution of* (4.26) *in* $C^+[0,1]$*.*

Remark 4.7. In [6, PP. 441–442] Karapetyants et al. studied the existence of positive solutions of the algebraic equation

$$\xi^m - d\xi - b = 0, \tag{4.27}$$

with $m > 0$, $m \neq 1$ and a, $b \in \mathbb{R} - \{0\}$. They investigated the positive solvability of (4.27) by using the properties of the function

$$f(\xi) = \xi^m - d\xi - b.$$

Set

$$c_0 := \left(\frac{d}{m}\right)^{\frac{1}{m-1}}, \quad E := f(c_0). \tag{4.28}$$

The authors of [6] obtained the following result which we state without proof.

Theorem 4.8. *Let $m > 0$, $m \neq 1$ and $a, b \in \mathbb{R} - \{0\}$. Let E and c_0 be as in* (4.28)*. Equation* (4.27)

(i) *does not have positive solutions if either $d < 0$, $b < 0$ or $d > 0$, $b < 0$, $m > 1$, $E > 0$ or $d > 0$, $b > 0$, $0 < m < 1$, $E < 0$;*

(ii) *has a unique positive solution*
   ii.1 $\xi = c_1 > 0$ *if* $d < 0$, $b > 0$;
   ii.2 $\xi = c_1 > c_0 > 0$ *if either* $d > 0$, $b < 0$, $0 < m < 1$ *or* $d > 0$, $b > 0$, $m = 1$;
   ii.3 $\xi = c_0 > 0$ *if* $E = 0$ $c_1 > c_0 > 0$ *if either and either* $d > 0$, $b < 0$, $m > 1$ *or* $d > 0$, $b > 0$, $0 < m < 1$

(iii) *has two positive solutions $\xi = c_2$ and $\xi = c_3$, $0 < c_2 < c_0 < c_3$, if either $d > 0$, $b < 0$, $m > 1$, $E < 0$ or $d > 0$, $b > 0$, $0 < m < 1$, $E > 0$.*

5. Asymptotic solution of linear equation in general case

In this section we investigate the special case $m = 1$ of (1.4) when $0 < \alpha < 1$. In other words, we give the asymptotic of the equation

$$\phi(x) = \frac{\lambda(x)}{\Gamma_q(\alpha)} x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt + f(x) \tag{5.1}$$

$$(0 < x < \infty, 0 < \alpha < 1).$$

In [2, P.214], the authors studied (1.4) for all $\alpha > 0$ when $\lambda(x) = \lambda$ for all $x \in (0, a]$, and $f \in \mathcal{L}_q^1[0, a]$ where a is a positive number satisfying the inequality

$$|\lambda| a^\alpha (1-q)^\alpha < 1.$$

They proved that the q-integral equation (5.1) under the previous conditions has a unique solution

$$\phi(x) = f(x) + \lambda x^{\alpha-1} \int_0^x (qt/x;q)_{\alpha-1} \varepsilon^{-q^\alpha t} e_{\alpha,\alpha}(\lambda t^\alpha; q) f(t)\, d_qt,$$

in the space $\mathcal{L}_q^1[0, a]$ where ε is the q-translation operator defined in (2.6).

Theorem 5.1. *Let $f(x)$ and $\lambda(x)$ have the asymptotics as $x \to 0$*

$$f(x) \sim \sum_{k=-1}^{\infty} f_k x^{\alpha k}, \tag{5.2}$$

and

$$\lambda(x) \sim \sum_{k=-1}^{\infty} \lambda_k x^{\alpha k}. \tag{5.3}$$

Assume that

$$\lambda_{-1} \neq \frac{\Gamma_q(\alpha k+\alpha+1)}{\Gamma_q(\alpha k+1)} \quad (k=-1,0,1,....). \tag{5.4}$$

Then the unique power asymptotic solution $\phi(x)$ of (5.1) near zero in the space of all continuous functions is given by the form $\phi(x) \sim \sum_{k=-1}^{\infty} \varphi_k x^{\alpha k}$, where φ_k is given by

(5.5)

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda_{-1}\right]^{-1} \times \left[\sum_{i=-1}^{k-1} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k\right]$$

$$(k=-1,0,1,2,......).$$

Proof. Using (3.11) we obtain

$$\Phi_{-1,k+1} = \sum_{i=-1}^{k} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)}\lambda_{k-i-1}\varphi_i + f_k \quad (k=-1,0,1,...).$$

Substitute with $p=-1$ in (3.4) yields

$$\phi(x) \sim \sum_{j=-1}^{\infty} \varphi_j x^{\alpha j}, \tag{5.6}$$

and from (3.2) with $p=-1$ and $m=1$

$$\phi(x) \sim x^{-\alpha} \sum_{k=0}^{\infty} \Phi_{-1,k} x^{\alpha k} = \sum_{j=-1}^{\infty} \Phi_{-1,j+1} x^{\alpha j}. \tag{5.7}$$

Compared to coefficients of $x^{\alpha j}$ in (5.6) and (5.7) we obtain

$$\Phi_{-1,j+1} = \varphi_j \quad (j=-1,0,1,....).$$

So, we have the following formulas for the coefficients φ_k

$$\varphi_k = \sum_{i=-1}^{k} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k \quad (k=-1,0,1,...),$$

equivalently

(5.8)

$$\left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda_{-1}\right]\varphi_k = \sum_{i=-1}^{k-1} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k \ (k=-1,0,1,...).$$

Hence if (5.4) satisfied, asymptotic solution $\phi(x)$ of equation (5.1) is given by the form (5.6) where φ_k given by (5.5). □

Theorem 5.2. *Let $f(x)$ and $\lambda(x)$ have the asymptotics* (5.2) *and* (5.3)*, respectively as $x \to 0$. Assume that there exists a number $j \in \{-1, 0, 1, ...\}$ such that*

$$\lambda_{-1} = \frac{\Gamma_q(\alpha j + \alpha + 1)}{\Gamma_q(\alpha j + 1)}. \tag{5.9}$$

If the coefficients $f_k\,(k = -1, 0, 1, ..., j)$ in the asymptotic expansion (5.2) *satisfy the relation*

$$\sum_{i=-1}^{j-1} \frac{\Gamma_q(\alpha j + 1)\lambda_{j-i-1}}{\Gamma_q(\alpha i + \alpha + 1)} \varphi_i + f_j = 0, \tag{5.10}$$

then the unique power asymptotic solution $\phi(x)$ of equation (5.1) *is given by*

$$\phi(x) \sim c x^{\alpha j} \sum_{\substack{k=-1 \\ k \neq j}}^{\infty} \varphi_k x^{\alpha k}, \tag{5.11}$$

where c is an arbitrary constant. If the condition (5.10) *is not satisfied, then equation* (5.1) *does not have any asymptotic solution of the form* (5.6).

Proof. Using Theorem 5.1 and suppose (5.4) is not valid. This means there exists a number $j \in \{-1, 0, 1, ...\}$ such that (5.9) holds. In this case the coefficients $f_k\,(k = -1, 0, 1, ..., j)$ in the asymptotic expansion (5.2) satisfy the relation (5.10), where $\varphi_i\,(i = -1, 0, 1, ..., j-1)$ are expressed via $f_i\,(i = -1, 0, 1, ..., j-1)$ by means of (5.5). For example, when $j = -1, 0, 1$, the relations (5.9) and (5.10) have the form

$$\lambda_{-1} = \frac{1}{\Gamma_q(1-\alpha)} \quad f_{-1} = 0 \quad \text{for} \quad j = -1$$

$$\lambda_{-1} = \Gamma_q(\alpha + 1) \quad \lambda_0 \varphi_{-1} + f_0 = 0 \quad \text{for} \quad j = 0$$

$$\lambda_{-1} = \frac{\Gamma_q(2\alpha + 1)}{\Gamma_q(\alpha + 1)} \quad \Gamma_q(\alpha + 1)\lambda_1 \varphi_{-1} + \lambda_0 \varphi_0 + f_1 = 0 \quad \text{for} \quad j = 1.$$

Thus if condition (5.10) is satisfied, then the asymptotic solution of (5.1) has the form (5.11), where c is an arbitrary constant and $\varphi_k\,(k \neq j)$ are given by (5.5). If condition (5.10) is not satisfied, equation (5.1) does not have any asymptotic solution of the form (5.6) □

Theorem 5.3. *Let*

$$f(x) \sim \sum_{k=0}^{\infty} f_k x^{\alpha k}, \tag{5.12}$$

and $\lambda(x)$ has the asymptotic (5.3), as $x \to 0$, and let condition (5.4) be satisfied. Then the unique power asymptotic solution $\phi(x)$ of equation (5.1) is given by the form

$$\phi(x) \sim \sum_{k=0}^{\infty} \varphi_k x^{\alpha k}, \tag{5.13}$$

where $\varphi_k\,(k \in \mathbb{N}_0)$ are given by

(5.14)

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k + 1)}{\Gamma_q(\alpha k + \alpha + 1)}\lambda_{-1}\right]^{-1} \times \left[\sum_{i=0}^{k-1} \frac{\Gamma_q(\alpha k + 1)\lambda_{k-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_k\right].$$

Proof. If $\lambda(x)$ has the asymptotic (5.2), then (5.8) takes the form

$$(1 - \Gamma_q(1-\alpha)\lambda_{-1})\varphi_{-1} = 0,$$

(5.15)

$$\left[1 - \frac{\Gamma_q(\alpha k + 1)}{\Gamma_q(\alpha k + \alpha + 1)}\lambda_{-1}\right]\varphi_k = \sum_{i=-1}^{k-1} \frac{\Gamma_q(\alpha k + 1)\lambda_{k-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_k \;(k = 0, 1, ...).$$

When condition (5.4) holds, $\varphi_{-1} = 0$ and hence the asymptotic solution (5.6) of equation (5.1) has the form (5.13), where $\varphi_k\,(k = 0, 1, 2, ...)$ are given by (5.14). □

Theorem 5.4. *Assume that the functions $f(x)$ and $\lambda(x)$ have the asymptotics (5.12) and (5.3), respectively, as $x \to 0$. If $\lambda_{-1} = 1$, then the unique power asymptotic solution $\phi(x)$ of equation (5.1) in the space $\mathcal{L}_q^1[0, a]$ for some $a > 0$ is given by*

$$\phi(x) \sim cx^{-\alpha} + \sum_{k=0}^{\infty} \varphi_k x^{\alpha k}, \tag{5.16}$$

where c is an arbitrary constant and $\varphi_k\,(k \in \mathbb{N}_0)$ are found from (5.14). Assume there exists a number $j \in \{-1, 0, 1, ...\}$ such that

$$(1 - \Gamma_q(1-\alpha)\lambda_{-1})\varphi_{-1} = 0 \quad (j = -1)$$

and

$$\sum_{i=-1}^{j-1} \frac{\Gamma_q(\alpha j + 1)\lambda_{j-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_j = 0 \quad (j \in \mathbb{N}_0). \tag{5.17}$$

If the coefficients $f_k\,(k = 0, 1, ..., j)$ in the asymptotic expansion (5.12) satisfy the relation

$$\sum_{i=0}^{k-1} \frac{\Gamma_q(\alpha k + 1)\lambda_{k-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_k = 0, \tag{5.18}$$

then the unique power asymptotic solution $\phi(x)$ *of equation* (5.1) *is given by*

$$\phi(x) \sim cx^{\alpha j} + \sum_{\substack{k=0 \\ k\neq j}}^{\infty} \varphi_k x^{\alpha k}, \tag{5.19}$$

where c *is an arbitrary constant.*

Proof. If condition (5.4) is not valid, then there exists a number $j \in \{-1, 0, 1, ...\}$ such that (5.9) holds. Then (5.10) has the form (5.17). When $\lambda_{-1} = 1$, the asymptotic solution $\phi(x)$ of equation (5.1) has the form (5.16), where c is an arbitrary constant and $\varphi_k\,(k = 0, 1, 2, ...)$ are found from (5.14). If $\lambda_{-1} \neq 1$ and (5.17) holds, then $\varphi_{-1} = 0$ and coefficients $f_k\,(k = 0, 1, ..., j)$ in the asymptotic expansion (5.12) satisfy the relation (5.18), where $\varphi_i\,(i = 0, 1, 2, ..., j-1)$ are expressed via $f_i\,(i = 0, 1, ..., j-1)$ by formulas (5.14). In this case the the asymptotic solution $\phi(x)$ of equation (5.1) has the form (5.19), where c is an arbitrary constant. □

Theorem 5.5. *Let the functions* $f(x)$ *and* $\lambda(x)$ *have asymptotic of the forms* (5.12) *and* (5.3)*, respectively. Then the unique power asymptotic solution* $\phi(x)$ *of equation* (5.1) *with any* $\alpha > 0$ *is given by* (5.13)*, where* φ_k $(k = 0, 1, ...)$ *are found from*

$$\varphi_{-1} = 0,\ \varphi_0 = f_0,\ \varphi_k = \sum_{i=0}^{k-1} \frac{\lambda_{k-1-i}\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha(i+1)+1)}\varphi_i + f_k. \tag{5.20}$$

Proof. This proof according to Theorem 5.4. □

Now, we use Theorem 5.5 to give asymptotics of $\phi(x)$ as $x \to 0$ when $f(x)$ has the asymptotic (5.12).

Corollary 5.6. *The asymptotic solution of the linear Volterra q-integral equation*

$$\phi(x) = \frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1}\phi(x)\, d_q t + f(x)\ (x > 0)$$

is given by

$$\phi(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma_q(\alpha n + 1)} \left[\sum_{k=0}^{n} f_k \lambda^{-k}\Gamma_q(\alpha k + 1)\right] x^{\alpha n}.$$

Proof. We apply Theorem 5.5 with $\lambda(x) = \lambda$. That is in (5.3)

$$\lambda_k = 0 \text{ for all } k \neq 0 \text{ and } \lambda_0 = 1.$$

Hence, the coefficients φ_k of the solution (5.13) satisfy the first order difference equation

$$\varphi_k - \frac{\lambda}{\Gamma_q(\alpha k+1)}\varphi_{k-1} = f_k \ (k\in\mathbb{N}), \quad \varphi_0 = f_0.$$

Set $\psi_k = \varphi_k \lambda^{-k}\Gamma_q(\alpha k+1)$ $(k \geq 1)$. Then ψ_k satisfies the difference equation

$$\psi_k - \psi_{k-1} = f_k\lambda^{-k}\Gamma_q(\alpha k+1).$$

Hence $\psi_n = \sum_{k=0}^{n} f_k\lambda^{-k}\Gamma_q(\alpha k+1)$. Consequently,

$$\varphi_n = \frac{\lambda^n}{\Gamma_q(\alpha n+1)}\sum_{k=0}^{n} f_k\lambda^{-k}\Gamma_q(\alpha k+1),$$

which proves the Corollary. □

Example 5.7. Equation (5.1) with $\lambda(x) = \lambda x^{\alpha(m-1)}$

$$\phi(x) = \frac{\lambda x^{\alpha m-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\, + f(x) \tag{5.21}$$

$(0 < x < \infty,\ 0 < \alpha < 1,\ m = 1, 2, ...;\ \lambda \neq 0)$ and $f(x)$ has the asymptotic (5.2). In this case $f(x)$ has the form

$$f(x) = f_{-1}x^{-\alpha} + f_0(x^{\alpha}),$$

$f_0(z) := \sum_{k=0}^{\infty} f_k z^k$ is an entire function in z^{α}. Hence,

$$\lambda_{m-1} = \lambda,\ \lambda_k = 0 \quad (k = -1, 0, 1, ...; k \neq m-1), \tag{5.22}$$

in (5.3) and therefore the relation (5.20) takes the form

$$\varphi_k = f_k \quad (k = -1, 0, 1, ..., m-2), \tag{5.23}$$

$$\varphi_k = \frac{\Gamma_q(\alpha k+1)\lambda}{\Gamma_q[\alpha(k-m+1)+1]}\varphi_{k-m} + f_k \quad (k = m-1, m, ...).$$

Thus we obtain for $k = -1, 0, 1, ..., m-2; n = 1, 2, ...$, that

$$\varphi_{nm+k} = \frac{\Gamma_q[\alpha(nm+k)+1]}{\Gamma_q[\alpha(nm+k-m+1)+1]}\lambda\varphi_{(n-1)m+k} + f_{nm+k}.$$

The asymptotic of solution $\phi(x)$ of equation (5.21)

$$\phi(x) \sim \sum_{k=-1}^{m-2}\sum_{n=0}^{\infty}\left[\sum_{j=1}^{n}\lambda^{n-j}\left(\prod_{i=j+1}^{n}\frac{\Gamma_q(\alpha(im+K)+1)}{\Gamma_q(\alpha(im-m+K+1)+1)}\right)\right]\times \frac{x^{\alpha(nm+k)}}{\Gamma_q(\alpha(nm+k)+1)}.$$

Example 5.8. The equation

$$\phi(x) = \frac{\lambda x^{\alpha m-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt \,+ \frac{d}{x^\alpha} + be_q(x^\alpha(1-q)) \tag{5.24}$$

$$(0 < x^\alpha(1-q) < 1; 0 < \alpha < 1; m = 1, 2, \ldots).$$

Hence,

$$f_{-1} = d,\ f_k = \frac{b}{\Gamma_q(k+1)}\,(k = 0, 1, 2, ...).$$

Consequently, equation (5.24) has the asymptotic solution, as $x \to 0$,

$$\begin{aligned}\phi(x) \;&\sim\; d\sum_{n=0}^{\infty}\lambda^n\prod_{i=0}^{n-1}\frac{\Gamma_q[\alpha(im-1)+1]}{\Gamma_q(\alpha im+1)}x^{\alpha(mn-1)}\\ &+\; b\sum_{k=-1}^{m-2}\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\frac{\lambda^{n-j}}{\Gamma_q(jm+k+1)}\prod_{i=j}^{n-1}\frac{\Gamma_q(\alpha(im+K)+1)}{\Gamma_q(\alpha(im+K+1)+1)}\right]x^{\alpha(nm+k)}.\end{aligned}$$

6. **Exact solutions of linear equations**

In the section we show that in some cases the asymptotic solution $\phi(x)$ of the linear equation (5.1) with certain conditions on $\lambda(x)$ and $f(x)$ gives the exact solution. This result is a q-analogue of the result introduced by Saigo and Kilbas in [9]. Consider (5.1) with $\lambda(x) = \lambda x^{-\alpha}$ and

$$f(x) = f_1 x^{-\alpha} + f_0(x^\alpha)$$

where $f_0(z) := \sum_{k=0}^{\infty} f_k z^k$ is an analytic function of z^α in a disk around zero, say $|z^\alpha| < R$. In this case we have the integral equation

$$\phi(x) = \frac{\lambda}{x\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt \,+ f(x^\alpha) \tag{6.1}$$

$$(0 < x^\alpha < R, 0 < \alpha < 1, \lambda \neq 0),$$

That is

$$\lambda_{-1} = \lambda,\ \lambda_k = 0\ \ (k \in \mathbb{N}_0)$$

in (5.3) and therefore the relation in (5.8) can be simplified to

$$\left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]\varphi_k = f_k \quad (k = -1, 0, 1, ...). \tag{6.2}$$

Condition (5.4) takes the form

$$\lambda \neq \frac{\Gamma_q(\alpha k+\alpha+1)}{\Gamma_q(\alpha k+1)} \quad (k = -1, 0, 1, ...). \tag{6.3}$$

Let (6.3) hold. Then from (6.2) we obtain

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} f_k \quad (k=-1,0,1,...),$$

and the asymptotic solution (5.6) has the form

$$\phi(x) \sim \sum_{k=-1}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} f_k x^{\alpha k} \text{ as } x \to 0. \tag{6.4}$$

Since

$$\left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} \sim (1-\lambda(1-q)^{\alpha})^{-1} \quad (k \to \infty),$$

the power series on the left hand side of (6.4) is an analytic function in z^{α} for $|z^{\alpha} < R|$ and the asymptotic solution give an exact solution. If (6.3) does not hold and there exists a number $j \in \{-1,0,1,...\}$ such that

$$\lambda = \frac{\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha j+1)}, \tag{6.5}$$

the condition of the asymptotic solvability (5.10) takes the simple form

$$f_j = 0, \tag{6.6}$$

and the asymptotic solution of equation (6.1) has the form

$$\phi(x) \sim c x^{\alpha j} + \sum_{\substack{k=-1\\ k\neq j}}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(\alpha j+1)}\right]^{-1} f_k x^{\alpha k}. \tag{6.7}$$

Similarly, the power series on the right hand side of (6.7) represents an analytic function in z^{α} for $|z^{\alpha}| < R$ and it is the exact solution in this case.

In the following examples we get exact solutions of (6.1) for certain choices of the function $f_0(x^{\alpha})$.

Example 6.1. The equation

$$\phi(x) = \frac{\lambda}{x\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt \, + b x^{\alpha l} \tag{6.8}$$

$$(0 < x < \infty,\ 0 < \alpha < 1,\ \lambda \neq \frac{\Gamma_q(\alpha l+\alpha+1)}{\Gamma_q(\alpha l+1)})$$

and $l \in \{-1,0,1,.....\}$ has the solution

$$\phi(x) = \left[1 - \frac{\Gamma_q(\alpha l+1)}{\Gamma_q(\alpha l+\alpha+1)}\lambda\right]^{-1} b x^{\alpha l}.$$

It is worth noting that the homogeneous equation

$$\frac{\Gamma_q(\alpha l+1)}{\Gamma_q(\alpha l+\alpha+1)}\phi(x)=\frac{1}{x\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt$$

$$(0<x<\infty, 0<\alpha<1),$$

for $l=-1,0,1,.....$ has the solution $\phi(x)=cx^{\alpha l}$ with c is an arbitrary constant.

Example 6.2. The equation

$$\phi(x)=\frac{\lambda}{x\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\;+\frac{d}{x^\alpha}+be_q(x^\alpha(1-q)) \tag{6.9}$$

$$(0<x^\alpha(1-q)<1, 0<\alpha<1),$$

$e_q(x^\alpha(1-q))=\sum_{j=0}^{\infty}\frac{x^{\alpha j}}{\Gamma_q(j+1)}$, has the solution

$$\phi(x)=\frac{d}{1-\Gamma_q(1-\alpha)\lambda}x^{-\alpha}+b\sum_{k=0}^{\infty}\left[1-\frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1}\frac{x^{\alpha k}}{\Gamma_q(k+1)}.$$

If (6.3) holds, the equation

$$\phi(x)=\frac{1}{x\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\;+be_q(x^\alpha(1-q))$$

$$(0<x^\alpha(1-q)<1, 0<\alpha<1),$$

and

$$\phi(x)=\frac{\Gamma_q(\alpha j+\alpha+1)}{x\Gamma_q(\alpha j+1)\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt+\frac{d}{x^\alpha}+b\left[e_q(x^\alpha(1-q))-\frac{x^{\alpha j}}{\Gamma_q(j+1)}\right]$$

$(0<x<\infty,\ 0<\alpha<1,\ j\in\{0,1,2,....\})$ have solution

$$\phi(x)=cx^{-\alpha}+b\sum_{k=0}^{\infty}\left[1-\frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(1-\alpha)}\right]^{-1}\frac{x^{\alpha k}}{\Gamma_q(k+1)},$$

and

$$\begin{aligned}\phi(x)=cx^{-\alpha j} &+ \frac{d\,\Gamma_q(\alpha j+1)x^{-\alpha}}{\Gamma_q(\alpha j+1)-\Gamma_q(1-\alpha)\Gamma_q(\alpha j+\alpha+1)}\\ &+ b\sum_{\substack{k=0\\k\neq j}}^{\infty}\left[1-\frac{\Gamma_q(\alpha k+1)\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(\alpha j+1)}\right]^{-1}\frac{x^{\alpha k}}{\Gamma_q(k+1)},\end{aligned}$$

respectively, where c is an arbitrary constant.

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