King Saud University: Mathematics Department Math-254
Second Semester 1443 H Solution (Midterm Exam.)
Maximum Marks = 30 Time: 120 mins.

Question 1: Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $x^2 - 2.6x = 2.31$ lying in the interval $[3.0, 3.5]$ using the bisection method. Find the approximations to the root with this degree of accuracy. [5 Marks]

Solution. Here $a = 3.0$, $b = 3.5$ and $k = 1$, then by using error bound formula of the bisection method, we get

$$n \geq \frac{\ln[10^1(3.5 - 3.0)]}{\ln 2} = 2.3219 \approx 3.$$

So no more than three iterations are required to obtain an approximation accurate to within 10^{-1}.
The given function $f(x) = x^2 - 2.6x - 2.31$ is continuous on $[3.0, 3.5]$, so starting with $a_1 = 3.0$ and $b_1 = 3.5$, we compute:

$$a_1 = 3.0: \quad f(3.0) = -1.1100 \quad \text{and} \quad b_1 = 3.5: \quad f(3.5) = 0.8400,$$

since $f(3.0)f(3.5) < 0$, so that a root of $f(x) = 0$ lies in the interval $[3.0, 3.5]$. Using bisection formula (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = \frac{3.0 + 3.5}{2} = 3.2500; \qquad f(c_1) = -0.1975.$$

Hence the function changes sign on $[c_1, b_1] = [3.2500, 3.5]$. To continue, we squeeze from right and set $a_2 = c_1$ and $b_2 = b_1$. Then the bisection formula gives

$$c_2 = \frac{a_2 + b_2}{2} = \frac{3.2500 + 3.5}{2} = 3.3750; \qquad f(c_2) = 0.3056.$$

Finally, the function changes sign on $[c_1, c_2] = [3.25, 3.375]$, gives

$$c_3 = \frac{a_3 + b_3}{2} = \frac{3.25 + 3.375}{2} = 3.3125,$$

the value of the third approximation which is accurate to within 10^{-1}.

Question 2: Find smallest interval $[a, b]$ with a and b are integers and $b = a+1$ such that the root $(25)^{1/3}$ lies in the interval. Use $x_0 = a$ to compute second approximation to the root by using Newton's formula. Show that the developed formula converges faster to the root. [5 Marks]

Solution. Let $x = (25)^{1/3}(= 2.9240)$ which gives $f(x) = x^3 - 25$ and let $a = 2$, then $b = 3$ and

$$f(2) = 2^3 - 25 = -17 \qquad \text{and} \qquad f(3) = 3^3 - 25 = 2,$$

so $f(2)f(3) < 0$. Hence we have the interval $[2, 3]$.
Since $f(x) = x^3 - 25$, so $f'(x) = 3x^2$. Using the Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 25}{3x_n^2} = g(x_n).$$

Now using $x_0 = 2$, we have

$$x_1 = x_0 - \frac{x_0^3 - 25}{3x_0^2} = 3.4167,$$

and

$$x_2 = x_1 - \frac{x_1^3 - 25}{3x_1^2} = 2.9916,$$

the second approximation.
The fixed-point form of Newton's formula for this problem is

$$g(x) = x - \frac{x^3 - 25}{3x^2},$$

and by taking the derivative, we have

$$g'(x) = 1 - \frac{(3x^2)(3x^2) - (x^3 - 25)6x}{3x^2} = \frac{(6x^3 - 150)}{9x^3},$$

and at $x = (25)^{1/3}$, we get

$$g'((25)^{1/3}) = \frac{(6((25)^{1/3})^3 - 150)}{9((25)^{1/3})^3} = 0.$$

Thus Newton's formula gives faster convergence to the root.

Question 3: Show that the x-value of the intersection point (x, y) of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$ is lying in the interval $[0.5, 1]$. Then use Secant method to find its second approximation, when $x_0 = 0.5$ and $x_1 = 0.55$. Also, find the intersection point. [5 Marks]

Solution. For the intersection of the graphs, we mean that $x^3 + 2x - 1 = \sin x$ and it gives, $x^3 + 2x - 1 - \sin x = 0$. Thus, $f(x) = x^3 + 2x - \sin x - 1$. Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5) = -0.3544$, $f(1.0) = 1.1585$, which shows that $f(0.5)f(1.0) < 0$. Hence the x-value (or root of $f(x) = 0$) lies in the interval $[0.5, 1.0]$. Applying Secant iterative formula to find the approximation of this root of the equation, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^3 + 2x_n - \sin x_n - 1)}{(x_n^3 + 2x_n - \sin x_n - 1) - (x_{n-1}^3 + 2x_{n-1} - \sin x_{n-1} - 1)}, \qquad n \geq 1.$$

Finding the first approximation using the initial approximations $x_0 = 0.5$ and $x_1 = 0.55$, we get

$$x_2 = 0.55 - \frac{(0.55 - 0.5)((0.55)^3 - 2(0.55) - \sin(0.55) - 1)}{((0.55)^3 - 2(0.55) - \sin(0.55) - 1) - ((0.5)^3 - 2(0.5) - \sin(0.5) - 1)} = 0.6806,$$

and the second approximation using the initial approximations $x_1 = 0.55$ and $x_2 = 0.6806$, we get

$$x_3 = 0.6806 - \frac{(0.6806 - 0.55)((0.0.6806)^3 - 2(0.6806) - \sin(0.6806) - 1)}{((0.0.6806)^3 - 2(0.6806) - \sin(0.6806) - 1) - (0.55 - 0.5)((0.55)^3 - 2(0.55) - \sin(0.55) - 1)},$$

So $x_3 = 0.6603$ is the second approximation of the x-value of the intersection point $(0.6603, 0.61)$.

Question 4: Use the simple Gaussian elimination method, find all values of k_1 and k_2 for which the following linear system is consistent or inconsistent. Find the solutions when the system is consistent. [5 Marks]

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & - & 3x_2 & + & k_1x_3 & = & 5 \\ 3x_1 & - & 4x_2 & + & 5x_3 & = & k_2 \end{array}$$

Solution. Writing the given system in the augmented matrix form

$$[A|b] = \left(\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 2 & -3 & k_1 & 5 \\ 3 & -4 & 5 & k_2 \end{array}\right) \equiv \left(\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & k_1-6 & -3 \\ 0 & 2 & -4 & k_2-12 \end{array}\right) \equiv \left(\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & (k_1-6) & -3 \\ 0 & 0 & (-2k_1+8) & k_2-6 \end{array}\right).$$

CASE I. Inconsistent system (no solution), if we take $k_1 = 4$ and $k_2 \neq 6$, gives

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (k_1-6)x_3 & = & -3 \\ & & & & (-2k_1+8)x_3 & = & (k_2-6) \end{array}$$

CASE II. Consistent system (infinitely many solutions), if we take $k_1 = 4$ and $k_2 = 6$, gives

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (k_1-6)x_3 & = & -3 \\ & & & & (-2k_1+8)x_3 & = & (k_2-6) \end{array}$$

gives

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (k_1-6)x_3 & = & -3 \\ & & & & 0x_3 & = & 0 \end{array}$$

Thus the infinitely many solutions

$$x_1 = -2+t, \quad x_2 = -3+2t, \quad x_3 = t, \quad t \in R.$$

CASE III. Consistent system (exactly one solution), if we take $k_1 \neq 4$ and $k_2 \in R$, gives

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (k_1-6)x_3 & = & -3 \\ & & & & (-2k_1+8)x_3 & = & (k_2-6) \end{array}$$

$x_1 = \dfrac{16k_1 + 9k_2 - 2k_1k_2 - 70}{-2k_1+8}, \quad x_2 = \dfrac{12k_1 + 6k_2 - k_1k_2 - 60}{-2k_1+8}, \quad x_3 = \dfrac{k_2-6}{-2k_1+8}$, the unique solution.

Question 5: Use LU decomposition by Dollittle's method to find the value(s) of α for which the following matrix

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{pmatrix},$$

is singular. Compute the unique solution of the linear system $A\mathbf{x} = [1, -1, -1]^T$ by using the largest negative integer value of α. [5 Marks]

Solution. Since we know that

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = 1 = l_{21}$, $m_{31} = \alpha = l_{31}$, and $m_{32} = -1 = l_{32}$, gives

$$A \equiv \begin{pmatrix} 1 & 1 & \alpha \\ 0 & \alpha - 1 & 1 - \alpha \\ 0 & 1 - \alpha & 1 - \alpha^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 & \alpha \\ 0 & \alpha - 1 & 1 - \alpha \\ 0 & 0 & 2 - \alpha^2 - \alpha \end{pmatrix} = U.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \alpha & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \alpha \\ 0 & \alpha - 1 & 1 - \alpha \\ 0 & 0 & 2 - \alpha^2 - \alpha \end{pmatrix} = LU,$$

which is the required decomposition of A. The matrix will be singular if

$$det(A) = det(U) = (1)(\alpha - 1)(2 - \alpha^2 - \alpha) = (\alpha - 1)(\alpha + 2)(1 - \alpha) = 0,$$

gives, $\alpha = 1$ and $\alpha = -2$.
To find the unique solution of the given system we take $\alpha = -1$ and it gives

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = LU.$$

To find unique solution we have to take $\alpha = -1$, and then solve the lower-triangular system

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \mathbf{b},$$

and it gives, $y_1 = 1$, $y_2 = -2$, $y_3 = -2$. Now solve the upper-triangular system

$$U\mathbf{x} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \mathbf{y},$$

and we obtained $\mathbf{x} = [0, 0, -1]^T$, the solution of the given system.

Question 6: Which method, Jacobi method or Gauss-Seidel method converges faster, for the solution of following system [5 Marks]

$$\begin{array}{rcrcrcr} 10x_1 & - & 2x_2 & + & x_3 & = & 9 \\ -2x_1 & + & 10x_2 & - & 2x_3 & = & 12 \\ -2x_1 & - & 5x_2 & + & 10x_3 & = & 18 \end{array}$$

If $x^{(0)} = [0,0,0]^T$, then using Gauss-Seidel method to find the number of iterations to get an accuracy within 10^{-4}.

Solution. Here we will show that the l_∞-norm of the Gauss-Seidel iteration matrix T_G is less than the l_∞-norm of the Jacobi iteration matrix T_J, that is

$$\|T_G\|_\infty < \|T_J\|_\infty.$$

The Jacobi iteration matrix T_J can be obtained from the given matrix A as follows

$$T_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & 1/5 & -1/10 \\ 1/5 & 0 & 1/5 \\ 1/5 & 1/2 & 0 \end{pmatrix},$$

Thus the l_∞-norm of the matrix T_J is

$$\|T_J\|_\infty = \max\left\{\frac{3}{10}, \frac{4}{10}, \frac{7}{10}\right\} = \frac{7}{10} = 0.7.$$

Similarly, Gauss-Seidel iteration matrix T_G can be obtained as

$$T_G = -(D+L)^{-1}U == \begin{pmatrix} 0 & 0.2 & -0.10 \\ 0 & 0.04 & 0.18 \\ 0 & 0.06 & 0.07 \end{pmatrix},$$

So the matrix form of Gauss-Seidel iterative method is and the l_∞-norm of the matrix T_G is

$$\|T_G\|_\infty = \max\{0.3, 0.22, 0.13\} = 0.3.$$

Since $\|T_G\|_\infty < \|T_J\|_\infty$, which shows that Gauss-Seidel method will converge faster than Jacobi method for the given linear system.
Since the Gauss-Seidel iterative method for the given system can be written as

$$\begin{array}{rcrcrcr} x_1^{(k+1)} & = & 0.1(9 & + & 2x_2^{(k)} & - & x_3^{(k)}) \\ x_2^{(k+1)} & = & 0.1(12 & + & 2x_1^{(k+1)} & + & 2x_3^{(k)}) \\ x_3^{(k+1)} & = & 0.1(18 & + & 2x_1^{(k+1)} & + & 5x_2^{(k+1)}) \end{array}$$

Starting with initial approximation $\mathbf{x}^{(0)} = [0,0,0]^T$, we get first approximation $\mathbf{x}^{(0)} = [0.9, 1.38, 2.67]^T$.
To find the number of iterations, we use the error bound formula as

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T_G\|^k}{1 - \|T_G\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \leq 10^{-4},$$

and it gives

$$\frac{(0.3)^k}{0.7}(2.67) \leq 10^{-4}.$$

Taking ln on both sides, we obtain, $k \geq 8.7619$, that is, $k = 9$.