
Question 1: (5 + 4 + 5)

Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Find the inverse of matrix A using **simple Gauss-elimination method** and then find the unique solution of the linear system.

(b) Construct **Jacobi iteration matrix** T_J and then compute error bound $\|\mathbf{x}-\mathbf{x}^{(\mathbf{20})}\|$, using the initial approximation $\mathbf{x}^{(0)} = [1, 1, 1]^T$ using the given above linear system.

(c) Develop the iterative formula $\quad x_{n+1} = \dfrac{x_n^2 - b}{2x_n - a}, \quad n \geq 0,$
for the approximate roots of the quadratic equation $x^2 - ax + b = 0$ using the Newton's method. Then use the formula to find the third approximation of the positive root of the equation $x^2 - 3x = 4$, starting with $x_0 = 3.5$.

Question 2: (4 + 4 + 4)

(a) Find approximation of the point of intersection of the graphs $y = x^3+2x-1$ and $y = \sin x$, using **bisection method** within accuracy 10^{-1} and the starting interval is $[0.5, 1.0]$.

(b) Let $p_2(x) = ax^2 + bx + c$ be the **quadratic Lagrange interpolating polynomial** for the data: $(1, 2), (2, 3), (3, \alpha)$. Compute the values of $a, b,$ and c. If $b = 1$, then find the value of α and the approximation of $f(2.5)$ using **linear Lagrange polynomial**.

(c) If $f(x) = \dfrac{2}{x}$, show that the **third divided difference** $f[1, 1, 1, 2] = -1$. Compute an error bound for the approximation of $f(1.5)$ using **cubic Newton's polynomial**.

Question 3: (5 + 4 + 5)

(a) Let $f(x) = x^3 + 1$ be defined in the interval $[0.1, 0.2]$. Use the error formula of two-point formula for the approximation of $f'(0.1)$ to find the unknown point $\eta \in (0.1, 0.2)$.

(b) Suppose that $f(0.25) = f(0.75) = \alpha$. Find α if **composite Trapezoidal rule** with $n = 2$ gives the value of the integral $\displaystyle\int_0^1 f(x)\, dx = 2$ and with $n = 4$ gives $\displaystyle\int_0^1 f(x)\, dx = 1.75$.

(c) Show that **Taylor's method of order 2** for the initial-value problem

$$e^y y' - e^x = 0, \quad 0 \leq x \leq 1, \quad y(0) = 1, \quad \text{with} \quad h = 0.5,$$

is

$$y_{i+1} = y_i + he^{(x_i - y_i)} \left[1 + \frac{h}{2}\left(1 - e^{(x_i - y_i)}\right)\right], \quad i \geq 0.$$

What are the values of y_0, y_1, y_2. Compare your approximate solution with the exact solution $y(x) = \ln(e^x + e - 1)$.

Question 1: (5 + 4 + 5)

Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Find the inverse of matrix A using **simple Gauss-elimination method** and then find the unique solution of the linear system.

Solution. Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let

$$AB = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the simple Gaussian elimination on the augmented matrix

$$[A|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right).$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1/3 & -1/3 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right).$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1/3 & -1/3 & 1 & 0 \\ 0 & 0 & 31/6 & 1/6 & -1/2 & 1 \end{array}\right).$$

We solve the first system

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & -1/3 \\ 0 & 0 & 31/6 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -1/3 \\ 1/6 \end{pmatrix},$$

by using backward substitution, we get

$$\begin{array}{rcrcr} 3b_{11} & & + \quad b_{31} & = & 1 \\ & 4b_{21} & - \quad 1/3b_{31} & = & -1/3 \\ & & 31/6b_{31} & = & 1/6 \end{array}$$

which gives $b_{11} = 10/31,\ b_{21} = -5/62,\ b_{31} = 1/31$. Similarly, the solution of the second linear system

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & -1/3 \\ 0 & 0 & 31/6 \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcr} 3b_{12} & & + \quad b_{32} & = & 0 \\ & 4b_{22} & - \quad 1/3b_{32} & = & 1 \\ & & 31/6b_{32} & = & -1/2 \end{array}$$

which gives $b_{12} = 1/31,\ b_{22} = 15/62,\ b_{32} = -3/31$. Finally, the solution of the third linear system

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & -1/3 \\ 0 & 0 & 31/6 \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcl} 3b_{13} & & + & b_{33} & = & 0 \\ & 4b_{23} & - & 1/3b_{33} & = & 0 \\ & & & 31/6b_{33} & = & 1 \end{array}$$

and it gives $b_{13} = -32/31,\ b_{23} = 1/62,\ b_{33} = 6/31$. Hence the elements of the inverse matrix B are

$$B = A^{-1} = \begin{pmatrix} 10/31 & 1/31 & -2/31 \\ -5/62 & 15/62 & 1/62 \\ 1/31 & -3/31 & 6/31 \end{pmatrix} = \begin{pmatrix} 0.3226 & 0.0323 & -0.0645 \\ -0.0806 & 0.2419 & 0.0161 \\ 0.0323 & -3/310.0968 & 0.1935 \end{pmatrix},$$

which is the required inverse of the given matrix A. To find the solution of the given system, we do as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10/31 & 1/31 & -2/31 \\ -5/62 & 15/62 & 1/62 \\ 1/31 & -3/31 & 6/31 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6/31 \\ 14/31 \\ 13/31 \end{pmatrix}.$$

Hence

$$x_1 = 0.1935, \qquad x_2 = 0.4516, \qquad x_3 = 0.4194,$$

is the solution of the given system.

(b) Construct **Jacobi iteration matrix** T_J and then compute error bound $\|\mathbf{x} - \mathbf{x}^{(\mathbf{20})}\|$, using the initial approximation $\mathbf{x}^{(0)} = [1, 1, 1]^T$ using the given above linear system.

Solution. Let

$$\begin{aligned} A &= \begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= L + D + U. \end{aligned}$$

Jacobi Method:

Since the Jacobi iteration matrix is defined as

$$T_J = -D^{-1}(L + U),$$

and by using the given information, we have

$$T_J = \begin{pmatrix} 0 & 0 & -1/3 \\ -1/4 & 0 & 0 \\ 0 & -2/5 & 0 \end{pmatrix}.$$

Then the l_∞ norm of the matrix T_J is

$$\|T_J\|_\infty = \frac{2}{5} = 0.4 < 1.$$

Thus the Jacobi method will converge for the given linear system.

(b) The Jacobi method for the given system is

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{3}\Big[1 \qquad\qquad - \; x_3^{(k)}\Big] \\ x_2^{(k+1)} &= \frac{1}{4}\Big[2 \; - \; x_1^{(k)} \qquad\qquad\Big] \\ x_3^{(k+1)} &= \frac{1}{5}\Big[3 \qquad\qquad - \; 2x_2^{(k)}\Big] \end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 1, x_2^{(0)} = 1, x_3^{(0)} = -1$, and for $k = 0$, we obtain the first approximations as

$$\mathbf{x^{(1)}} = [\mathbf{0}, \mathbf{1/4}, \mathbf{1/5}]^{\mathbf{T}}.$$

Using the error bound formula, we obtain

$$\|\mathbf{x} - \mathbf{x^{(20)}}\| \leq \frac{(0.4)^{20}}{1-0.4} \left\| \begin{pmatrix} 0 \\ 1/4 \\ 1/5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| \leq 1.8 \times 10^{-8}.$$

(c) Develop the iterative formula $\qquad x_{n+1} = \dfrac{x_n^2 - b}{2x_n - a}, \qquad n \geq 0,$

for the approximate roots of the quadratic equation $x^2 - ax + b = 0$ using the Newton's method. Then use the formula to find the third approximation of the positive root of the equation $x^2 - 3x = 4$, starting with $x_0 = 3.5$.

Solution.(1c) Given

$$f(x) = x^2 - ax + b,$$

therefore, we have

$$f(x_n) = x_n^2 - ax_n + b \quad \text{and} \quad f'(x_n) = 2x_n - a.$$

Using these functions values in the Newton's iterative formula, we have

$$x_{n+1} = x_n - \frac{x_n^2 - ax_n + b}{2x_n - a} = \frac{x_n^2 - b}{2x_n - a}, \qquad n \geq 0.$$

Finding the first three approximations of the positive root of $x^2 - 3x = 4$ using the initial approximation $x_0 = 3.5$ and $a = 3, b = -4$, we use the above formula by taking $n = 0, 1, 2$ as follows

$$x_1 = \frac{x_0^2 - b}{2x_0 - a} = 4.0625, \quad x_2 = \frac{x_1^2 - b}{2x_1 - a} = 4.0008, \quad x_3 = \frac{x_2^2 - b}{2x_2 - a} = 4.0000,$$

are the possible three approximations. Note that the positive root of $x^2 - 3x - 4 = 0$ is 4, so we have

$$|4 - x_3| = |4 - 4| = 0.0000,$$

the possible absolute error.

Question 2: (4 + 4 + 4)

(a) Find approximation of the point of intersection of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$, using **bisection method** within accuracy 10^{-1} and the starting interval is $[0.5, 1.0]$.

Solution. The graphs in the Figure **??** show that there is an intersection at about point $(0.66, 0.61)$. Using the function $f(x) = x^3 + 2x - \sin x - 1$ and the starting interval $[0.5, 1.0]$, we compute:

$$\begin{array}{rcll} a_1 & = & 0.5: & f(a_1) = -0.3544, \\ b_1 & = & 1.0: & f(b_1) = 1.1585. \end{array}$$

Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5).f(1.0) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1.0]$. Using formula (**??**) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \qquad f(c_1) = 0.240236.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the midpoint is:

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \qquad c_3 = 0.6875.$$

(b) Let $p_2(x) = ax^2 + bx + c$ be the **quadratic Lagrange interpolating polynomial** for the data: $(1, 2), (2, 3), (3, \alpha)$. Compute the values of a, b, and c. If $b = 1$, then find the value of α and the approximation of $f(2.5)$ using **linear Lagrange polynomial**.

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

using the given data points, we get

$$f(x) = p_2(x) = L_0(x)(2) + L_1(x)(3) + L_2(x)(\alpha),$$

where the Lagrange coefficients can be calculate as follows:

$$\begin{array}{rcl} L_0(x) & = & \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \dfrac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} = \dfrac{1}{2}(x^2 - 5x + 6), \\[2ex] L_1(x) & = & \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \dfrac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} = -(x^2 - 4x + 3), \\[2ex] L_2(x) & = & \dfrac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \dfrac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)} = \dfrac{1}{2}(x^2 - 3x + 2). \end{array}$$

Thus

$$f(x) = p_2(x) = \frac{1}{2}(x^2 - 5x + 6)(2) - (x^2 - 4x + 3)(3) + \frac{1}{2}(x^2 - 3x + 2)(\alpha).$$

Separating the coefficients of x^2, x and constant term, we get

$$f(x) = p_2(x) = \left(-2 + \frac{\alpha}{2}\right)x^2 + \left(7 - \frac{3\alpha}{2}\right)x + (-3 + \alpha).$$

Since the given value of the constant term is 5, using this, we get

$$\left(7 - \frac{3\alpha}{2}\right) = 1, \quad \text{gives} \quad \alpha = 4.$$

Thus by using $\alpha = 4$ and $x_0 = 2, x_1 = 3, x = 2.5$, we have

$$f(2.5) \approx p_1(2.5) = \frac{(2.5-3)}{(2-3)}(3) + \frac{(2.5-2)}{(3-2)}(4) = 3.5,$$

the required approximation of the function.

(c) If $f(x) = \dfrac{2}{x}$, show that the **third divided difference** $f[1,1,1,2] = -1$. Compute an error bound for the approximation of $f(1.5)$ using **cubic Newton's polynomial**.

Solution. We can find value of $f[1,1,1,2]$ by using divide difference as follows:

$$\begin{aligned} f[1,1,1,2] &= \frac{f[1,1,2] - f[1,1,1]}{2-1} = f[1,1,2] - f[1,1,1] \\ &= \frac{f[1,2] - f[1,1]}{2-1} - \frac{f''(1)}{2!} \\ &= \frac{f(2) - f(1)}{2-1} - \frac{f'(1)}{1!} - \frac{f''(1)}{2!} \\ &= f(2) - f(1) - f'(1) - \frac{f''(1)}{2}. \end{aligned}$$

Since $f(x) = \dfrac{2}{x}$, so we have, $f'(x) = -\dfrac{2}{x^2}$ and $f''(x) = \dfrac{4}{x^3}$. Thus

$$f[1,1,1,2] = f(2) - f(1) - f'(1) - \frac{f''(1)}{2} = 1 - 2 + 2 - 2 = -1,$$

is the required value.

Question 3: (5 + 4 + 5)

(a) Let $f(x) = x^3 + 1$ be defined in the interval $[0.1, 0.2]$. Use the error formula of two-point formula for the approximation of $f'(0.1)$ to find the unknown point $\eta \in (0.1, 0.2)$.

Solution. Since the exact value of the first derivative of the function at $x_0 = 0.2$ is

$$f'(x) = 3x^2 \quad \text{and} \quad f'(0.1) = 3(0.1)^2 = 0.03,$$

and the approximate value of $f'(0.1)$ using two point formula is

$$f'(0.1) \approx \frac{f(0.2) - f(0.1)}{0.1} = 0.07,$$

so error E can be calculated as

$$E = 0.03 - 0.07 = -0.04.$$

Using the error formula and $f''(\eta) = 6\eta$, we have

$$-0.04 = -\frac{0.1}{2} 6\eta,$$

and solving for η, we get $\eta = 0.1333$.

(b) Suppose that $f(0.25) = f(0.75) = \alpha$. Find α if **composite Trapezoidal rule** with $n = 2$ gives the value of the integral $\int_0^1 f(x)\ dx = 2$ and with $n = 4$ gives $\int_0^1 f(x)\ dx = 1.75$.

Solution. For $n = 2$, using the formula and $h = 0.5$, we have

$$\int_0^1 f(x)\ dx = 2 \approx T_2(f) = \frac{0.5}{2}\Big[f(0) + 2f(0.5) + f(1)\Big],$$

which is equivalent to

$$8 \approx f(0) + 2f(0.5) + f(1).$$

For $n = 4$, using the formula and $h = 0.25$, we have

$$\int_0^1 f(x)\ dx = 1.75 \approx T_4(f) = \frac{0.25}{2}\Big[f(0) + 2(2\alpha) + 2f(0.5) + f(1)\Big],$$

which is equals to

$$8(1.75) \approx f(0) + 2f(0.5) + f(1) + 4\alpha, \quad \text{or} \quad 8(1.75) \approx 8 + 4\alpha.$$

(using $8 \approx f(0) + 2f(0.5) + f(1)$). Solving for α, we get $\alpha \approx 1.5$, the required value.

(c) Show that **Taylor's method of order 2** for the initial-value problem

$$e^y y' - e^x = 0, \quad 0 \leq x \leq 1, \quad y(0) = 1, \quad \text{with} \quad h = 0.5,$$

is

$$y_{i+1} = y_i + he^{(x_i - y_i)}\left[1 + \frac{h}{2}\Big(1 - e^{(x_i - y_i)}\Big)\right], \quad i \geq 0.$$

What are the values of y_0, y_1, y_2. Compare your approximate solution with the exact solution $y(x) = \ln(e^x + e - 1)$.

Solution. Since the Taylor's method of order 2 is

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i), \quad \text{for} \quad i \geq 0,$$

and the given function is $f(x, y) = e^{x-y}$ with its first derivative $f'(x, y) = e^{x-y}[1 - e^{x-y}]$. So using these values, we have

$$y_{i+1} = y_i + he^{x_i - y_i} + \frac{h^2}{2}\left[e^{x-y}\Big(1 - e^{x-y}\Big)\right], \quad \text{for} \quad i \geq 0,$$

or

$$y_{i+1} = y_i + he^{(x_i - y_i)} \left[1 + \frac{h}{2} \left(1 - e^{(x_i - y_i)} \right) \right], \quad i \geq 0.$$

Now for $i = 0$, we have

$$y_1 = y_0 + he^{(x_0 - y_0)} \left[1 + \frac{h}{2} \left(1 - e^{(x_0 - y_0)} \right) \right],$$

and using $x_0 = 0, y_0$ and $h = 0.5$, we get y_1 as follows

$$y_1 = 1 + (0.5)e^{(0-1)} \left[1 + \frac{0.5}{2} \left(1 - e^{(0-1)} \right) \right] = 1.2130.$$

Similar way, we have the value of y_2 for taking $i = 1$, as follows

$$\begin{aligned} y_2 &= y_1 + he^{(x_1 - y_1)} \left[1 + \frac{h}{2} \left(1 - e^{(x_1 - y_1)} \right) \right] \\ &= 1.2130 + (0.5)e^{(0.5-1.2130)} \left[1 + \frac{0.5}{2} \left(1 - e^{(0.5-1.2130)} \right) \right] = 1.4893. \end{aligned}$$

the required approximation of $y(x)$ at $x = 1$. Thus

$$|y(1) - y_2| = |1.4899 - 1.4893| = 0.0006,$$

is the possible absolute error in our approximation.