problems for chapter I :-

problem 1-1 :-
Find the least upper bound and the greatest lowe bound of A_1, A_2 defined by setting

$$A_1 = \left\{ 2(-1)^{n+1} + (-1)^{\frac{n(n+1)}{2}} \left(2 + \frac{3}{n}\right), n \in \mathbb{N} \right\}$$

$$A_2 = \left\{ \frac{n-1}{n+1} \cos \frac{2n\pi}{3}, n \in \mathbb{N} \right\}.$$

problem ~~2~~ 2-1 :-
Find the sepremium and the infimum of the set

$$\Omega = \{0.2, 0.22, 0.222, \dots\}.$$

problem 3-1 :-
Find the greatest lower and the least upper bounds of the set of numbers

$$\frac{(n+1)^2}{2^n}, n \in \mathbb{N}.$$

problem 4-1 :-
Determine the least upper and the greatest lower bounds of the following sets :-

$$A = \left\{ \frac{m}{n}, m, n \in \mathbb{N}, m < 2n \right\}$$

$$B = \left\{ \sqrt{n} - [\sqrt{n}], n \in \mathbb{N} \right\}.$$

problem 5-1 :- Find

1) $\sup \{x \in \mathbb{R}, x^2 + x + 1 > 0\}$.
2) $\inf \{z = x + x^{-1}, x > 0\}$.
3) $\inf \{z = 2^x + 2^{\frac{1}{x}}, x > 0\}$.

problem 6-1:
Find the supremum and the infimum of the following sets:

1) $A = \left\{ \frac{m}{n} + \frac{4n}{m} , m, n \in \mathbb{N} \right\}$,

2) $B = \left\{ \frac{mn}{4m^2+n^2} , m \in \mathbb{Z}, n \in \mathbb{N} \right\}$,

3) $C = \left\{ \frac{m}{m+n} , m, n \in \mathbb{N} \right\}$,

4) $D = \left\{ \frac{m}{|m|+n} , m \in \mathbb{Z}, n \in \mathbb{N} \right\}$

5) $E = \left\{ \frac{mn}{1+m+n} , m, n \in \mathbb{N} \right\}$.

problem 7-1:
Find the sup, inf, Max, Min of each of the following:

1) $A = \left\{ \frac{1}{n} , n \in \mathbb{Z} , n \neq 0 \right\}$

2) $B = \{ x , 0 \leqslant x \leqslant \sqrt{2}$ and x is rational $\}$

3) $C = \{ x , x < 0$ and $x^2 - x - 1 < 0 \}$

problem 8-1:
1) Let $A = \{ x , x < \alpha \}$. prove the following:

i) If x is in A and $y < x$, then y is in A

ii) $A \neq \phi$

iii) $A \neq \mathbb{R}$

(v) If x is in A, then there is some number x' in A such that $x < x'$.

2) Suppose conversely that A satisfies (i)-(iv). prove that $A = \{ x , x < \sup A \}$.

solutions :
problem 1.1:

For $n = 1, 2, 3$, we have the terms
$-3 , -\frac{11}{2} , 5$

For $n \geqslant 4$, let us distinguish four cases
$n = 4k$, $n = 4k+1$, $n = 4k+2$ and $n = 4k+3$
Thus, we obtain respectively the terms :

$$\frac{3}{4k} , -\frac{3}{4k+1} , -4-\frac{3}{4k+2} , 4+\frac{3}{4k+3} , k \in \mathbb{N}$$

So

$$A_1 = \left\{-3, -\frac{11}{2}, 5\right\} \cup \left\{\frac{3}{4k}, -\frac{3}{4k+1}, -4-\frac{3}{4k+2}, 4+\frac{3}{4k+3}, k\in \mathbb{N}\right\}.$$

Hence, $\sup A_1 = 5$ and $\inf A_1 = -\frac{11}{2}$.

For A_2, let us put $n = 3k$, $n = 3k-1$, $n = 3k-2$ for $k \in \mathbb{N}$.

Thus $\cos \frac{2(3k)\pi}{3} = 1$, $\cos \frac{2(3k-1)}{3}\pi = -\frac{1}{2}$ and

$\cos \frac{2(3k-2)\pi}{3} = -\frac{1}{2}$, $k \in \mathbb{N}$.

Hence we see that

$$A_2 = \left\{\frac{3k-1}{3k+1}, -\frac{3k-2}{6k}, -\frac{3k-3}{2(3k-1)}, k \in \mathbb{N}\right\}$$

Therefore $\sup A_2 = \lim\limits_{k\to\infty} \frac{3k-1}{3k+1} = 1$ (increasing)

and $\inf A_2 = -\frac{1}{2} = \lim\limits_{k\to\infty} -\frac{3k-2}{6k}$
$= \lim -\frac{3k-3}{2(3k-1)}$.

problem 2_1:

clearly $\sup \Omega = 0.2222\ldots = \frac{2}{9}$

as $0.99999\ldots \approx 1$

$\xrightarrow{\text{division by } 9} 0.11111\ldots \approx \frac{1}{9} \Rightarrow \frac{2}{9} = 0.9999\ldots$

and $\inf \Omega = 0.2.$

problem 3_1:

By induction we can show that

$$2^n > (n+1)^3 \text{ for } n \geqslant 11.$$

Thus $0 < \frac{(n+1)^2}{2^n} < \frac{(n+1)^2}{(n+1)^3} = \frac{1}{n+1}$ for $n \geqslant 11$.

Therefore 0 is the greatest lower bound.

On the other hand, we can show that

$$2^n > (n+1)^2 \text{ for } n \geqslant 6.$$

Hence $\frac{(n+1)^2}{2^n} < 1$ for $n \geqslant 6$. The terms $2, \frac{9}{4}, \frac{25}{16}, \frac{36}{32}$ greater than 1 each.

Thus the least upper bound is $\frac{9}{4}$.

problem 4_1:

1) $m < 2n \Rightarrow \frac{m}{n} < 2$, whence 2 is an upper bound to show that it is the smallest, let $\varepsilon > 0$, then for any $N > [\frac{2}{\varepsilon}]$ we have $\frac{2(N-1)}{N} > 2 - \varepsilon$ (and $\frac{2(N-1)}{N} \in A$. $(N > [\frac{2}{\varepsilon}] \to \frac{2}{N} < \varepsilon \to 2 - \frac{2}{N} > 2 - \varepsilon)$

Also 0 is the inf because: given $\varepsilon > 0$ there is N such that $\frac{1}{N} < \varepsilon$, and $\frac{1}{N} \in A$.

2) We know that $0 \leq \sqrt{n} - [\sqrt{n}] < 1$.
Taking $n = k^2$, $k \in \mathbb{N}$, we see that $0 \in B$.
Thus $\inf B = 0$, as all elements of B are positive.
To show that $\sup B = 1$, observe first that
$[\sqrt{n^2+2n}] = n$, as $n \leq \sqrt{n^2+2n} < n+1$, $\forall n \in \mathbb{N}$.
For $0 < \varepsilon < 1$, a simple calculation
leads to

$$\sqrt{n^2+2n} - \left[\sqrt{n^2-2n}\right] = \sqrt{n^2+2n} - n$$

$$= n\left(\sqrt{1+\frac{2}{n}} - 1\right) = n\left(\sqrt{1+\frac{2}{n}} - 1\right)\frac{\sqrt{1+\frac{2}{n}}+1}{\sqrt{1+\frac{2}{n}}+1}$$

$$= n\,\frac{1+\frac{2}{n}-1}{\sqrt{1+\frac{2}{n}}+1} = \frac{2}{\sqrt{1+\frac{2}{n}}+1}$$

Thus

$$\sqrt{n^2+2n} - \left[\sqrt{n^2-2n}\right] = \frac{2}{\sqrt{1+\frac{2}{n}}+1} > 1-\varepsilon$$

is satisfied for any integer $n > \frac{(1-\varepsilon)^2}{2\varepsilon}$.

(For $\varepsilon \geq 1$, this is always satisfied $\forall n \in \mathbb{N}$).

problem 5_1:

1) $\sup\{x \in \mathbb{R},\ x^2+x+1 > 0\} = \infty$

and this is clear. since $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4} > 0$
for all values of x. Hence the sup is ∞.

2) $\inf\{z = x + \frac{1}{x},\ x > 0\} = 2$ because these
are the values of the function $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = 0 \Rightarrow x = \pm 1$

$\Rightarrow f(1) = 1 + \frac{1}{1} = 2$ is inf and for $x > 0$.

$f''(x) = \frac{2}{x^3} \Rightarrow f''(1) = 2 > 0$ (local min)

3) $\inf\{z = 2^{x} + 2^{\frac{1}{x}}, x > 0\} = 4$.

For, observe $\frac{a+b}{2} \geqslant \sqrt{ab}$, $a, b > 0$.
Therefore
$$\frac{2^{x} + 2^{\frac{1}{x}}}{2} \geqslant \sqrt{2^{\frac{1}{x}+x}} \geqslant \sqrt{2^{2}} = 2$$
with equality if and only if $x = 1$.

problem 6-1:

1) Using the inequality $\frac{a+b}{2} \geqslant \sqrt{ab}$ for $a, b > 0$
we get
$$\frac{m}{n} + \frac{4n}{m} \geqslant 4$$
with equality for $m = 2n$, $\left(a = \frac{2m}{n}, b = \frac{8n}{m}\right)$
Therefore $\inf A = 4$.
on the other hand this set is not bounded above, for instance $m = 1$ and n very large
So $\sup A = \infty$.

2) use $(a+b)^2 \geqslant 0$ and $(a-b)^2 \geqslant 0$ for
$a = 2m$ and $b = n$ to obtain
$$-\frac{1}{4} \leqslant \frac{mn}{4m^2+n^2} \leqslant \frac{1}{4}$$
with equalities for $m = -2n$ and $m = 2n$ respectively. Hence we see that
$$\inf B = -\frac{1}{4} \text{ and } \sup B = \frac{1}{4}.$$

3) We observe that $0 < \frac{m}{m+n} < 1$ and for any $\varepsilon > 0$ there exist positive integers N and M such that $\frac{1}{N+1} < \varepsilon$ and $\frac{M}{M+1} > 1 - \varepsilon$, whence $\inf C = 0$ and $\sup C = 1$

4) Since $-1 < \frac{m}{|m|+n} < 1$ for $m \in \mathbb{Z}, n \in \mathbb{N}$ we see that $\inf D = -1$ and $\sup D = 1$.
(as in (3)).

5) This set is not bounded above since for $m = n$ we get $\frac{n^2}{1+2n} \longrightarrow \infty$ as $n \longrightarrow \infty$.
Thus $\sup E = \infty$.
on the other hand, $mn \geqslant 1$, $mn \geqslant m$, $mn \geqslant n$
$\Rightarrow 3mn \geqslant 1 + m + n \Rightarrow \frac{mn}{1+m+n} \geqslant \frac{1}{3}$
with equality for $m = n = 1$. Hence, we see that $\inf E = \frac{1}{3}$.

problem 7-1:

1) $\sup A = 1$ and $\inf A = -1$.

2) $\inf A = 0$ and $\sup A = \sqrt{2}$.

3) $x^2 - x - 1 = 0$, $\Delta = (-1)^2 - 4(-1)(1) = 5$
$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$

x	$-\infty$		$\frac{1-\sqrt{5}}{2}$		$\frac{1+\sqrt{5}}{2}$		∞
x^2-x-1		$+$		$-$		$+$	

Thus $C = \left(\frac{1-\sqrt{5}}{2}, 0\right)$

$\Rightarrow \sup C = 0$, $\inf C = \frac{1-\sqrt{5}}{2}$

problem 8-1.

1) i) $x \in A \Rightarrow x < \alpha \Rightarrow y < x < \alpha \Rightarrow y < \alpha$
So $y \in A$.

ii) $\alpha - 1 < \alpha \Rightarrow \alpha - 1 \in A \Rightarrow A \neq \phi$.

iii) $\alpha + 1 \notin A$, so $A \neq \mathbb{R}$.

iv) If $x \in A$, then $x < \alpha$. Set $x' = \frac{x+\alpha}{2}$.
Thus $x < \frac{x+\alpha}{2} = x' < \alpha. \Rightarrow x' \in A$.

2) From (iii) there is some $y \notin A$.
If $y < x$, then x cannot be in A, because (i) would imply that $y \in A$; so $x < y$ and thus y is an upper bound for A.
Since $A \neq \phi$ by (ii), $\sup A$ exists.
Given x in A and choose x' in A with $x < x'$ by (iv). Then $x < x' \leq \sup A$
So $x < \sup A$.
Conversely, if $x < \sup A$, then there is some y in A with $x < y$. Hence $x \in A$ by (i).

problems for chapter II :

problem 1-2 :
Verify each of the following limits:

1) $\lim \dfrac{n+3}{n^3+4} = 0$

2) $\lim\limits_{n \to \infty} \dfrac{n!}{n^n} = 0$

3) $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

4) $\lim\limits_{n \to \infty} \sqrt[n]{a^n + b^n} = \max(a, b)$, $a, b \geq 0$.

problem 2-2 :
Find the following limits :

1) $\lim\limits_{n \to \infty} \dfrac{n}{n+1} - \dfrac{n+1}{n}$

2) $\lim\limits_{n \to \infty} n - \sqrt{n+a}\sqrt{n+b}$.

3) $\lim\limits_{n \to \infty} \dfrac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}}$

4) $\lim\limits_{n \to \infty} \dfrac{(-1)^n \sqrt{n} \sin(n^n)}{n+1}$

5) $\lim\limits_{n \to \infty} \dfrac{a^n - b^n}{a^n + b^n}$.

6) $\lim\limits_{n \to \infty} n c^n$, $|c| < 1$.

7) $\lim\limits_{n \to \infty} \dfrac{2^{n^2}}{n!}$

problem 3-2 :
1) prove that if a subsequence of a Cauchy sequence

converges, then so does the original Cauchy sequence.

2) prove that any subsequence of a convergent sequence converges.

problem 4-2:

Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n} \quad , \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

1) prove that the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

2) prove that they have the same limit.

problem 5-2:

Evaluate the limits:

1) $$\lim_{n \to \infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \dots + \sqrt[n]{e^{2n}}}{n}$$

2) $$\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \right)$$

3) $$\lim_{n \to \infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+n^2} \right).$$

problem 6-2:

1) show that if $c \neq 1$, then

$$c^m + c^{m+1} + \dots + c^n = \frac{c^m - c^{n+1}}{1-c}$$

2) Suppose that $|c| < 1$. prove that

$$\lim_{m, n \to \infty} c^m + \dots + c^n = 0.$$

3) Suppose that $\{x_n\}$ is a sequence with $|x_n - x_{n+1}| \leq c^n$, where $c < 1$.

prove that $\{x_n\}$ is a Cauchy sequence.

Answers for chapter II problems:

Pb 2-2:

1) 0
2) $-\frac{a+b}{2}$
3) $\frac{1}{2}$
4) 0
5) 0 if $a=b$, not exist if $a=-b$, 1 if $|a|>|b|$, -1 if $|a|<|b|$
6) 0
7) ∞

Pb 3-2:

1) Let $\{a_n\}$ be a cauchy sequence, and suppose that $\lim_{n\to\infty} a_n = l$. For any $\varepsilon>0$, choose J so that $|l-a_{n_j}|<\varepsilon/2$ for $j>J$. Then choose N so that $|a_n-a_m|<\varepsilon/2$ for $n, m>N$. Let $N_0=\max(N, n_j)$. If $n>N_0$, then $|a_n-a_{n_{J+1}}|<\varepsilon/2$ and $|a_{n_{J+1}}-l|<\varepsilon/2$. Consequently $|a_n-l|<\varepsilon/2$.

2) Suppose $\lim_{n\to\infty} a_n = l$ and $\{a_{n_j}\}$ a subsequence of $\{a_n\}$. If $\varepsilon>0$, then there is some N such that $|l-a_n|<\varepsilon$ for $n>N$.
Since $n_1<n_2<\dots$, there is some J such that $n_j>N$ for $j>J$. Thus $|l-a_{n_j}|<\varepsilon$ for $j<J$.
So $\lim_{j\to\infty} a_{n_j}=l$

Pb 4 - 2 +

1) use the inequality $\sqrt{ab} < \frac{a+b}{2}$ to show that $a_1 < a_n < a_{n+1} < b_{n+1} < b_n < b_1$

Thus $\{a_n\}$ is increasing and bounded by b_1 and $\{b_n\}$ is decreasing and bounded by a_1.

2) $\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{a_n}\sqrt{b_n}$

$\Longrightarrow \quad l = \sqrt{l}\sqrt{m} \implies l = m.$

Pb 5 - 2 +

1) $\int_0^2 e^x dx = e^2 - 1.$

2) Since $\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2} \leqslant n \cdot \frac{1}{n^2} \leqslant \frac{1}{n}$ we see that the limit is 0.

3) $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$

Pb 6 - 2 +

1) $c^n + c^{n+1} + \ldots + c^m = c^n(1 + c + \ldots + c^{m-n})$

$$= \frac{c^n(1 - c^{m-n+1})}{1-c} = \frac{c^n - c^{m+1}}{1-c}.$$

2) Since $|c| < 1$, we have $\lim_{n \to \infty} c^n = \lim_{m \to \infty} c^{m+1} = 0$

3) $|x_n - x_m| = |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \ldots + (x_{m-1} - x_m)|$

$\leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots + |x_{m-1} - x_m|$

$\leqslant c^n + \ldots + c^{m-1},$

So $\lim_{m,n \to \infty} |x_n - x_m| = 0$ by ②.

problems for chapter III:

problem 1-3:

Decide whether the following series is convergent or divergent:

1) $\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$

2) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

3) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

4) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-1}}$

5) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

6) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

7) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

problem 2-3:

A sequence $\{a_n\}$ is called Cesaro Summable with Cesaro sum l if

$$\lim_{n \to \infty} \frac{S_1+S_2+\cdots+S_n}{n} = l$$

where $S_k = a_1 + a_2 + \cdots + a_k$.

1) Show that a summable sequence ($\sum a_n$ converges) is Cesaro summable, with sum equals to its Cesaro sum.

2) Find a sequence which is not summable, but which is Cesaro summable.

problem 3-3:

Suppose that $a_n > 0$ and $\{a_n\}$ is Cesaro summable. suppose also that the sequence $\{n a_n\}$ is bounded. prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

problem 4-3:-
prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

problem 5-3:-
1) show that if $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
2) prove that if $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$ converges for any $\alpha > \frac{1}{2}$.

problem 6-3:-
Suppose that $\{a_n\}$ is decreasing and each $a_n \geqslant 0$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n \to \infty} n a_n = 0$.

problem 7-3:-
prove that if $a_n \geqslant 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, the $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.

problem 8-3:-
Show that $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ $\begin{cases} \text{converges if } a < e \\ \text{diverges if } a > e \\ \text{diverges also for } a = e. \end{cases}$
$a > 0$.
Same question for $\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$.

Answers to chapter III problems :

Pb 1-3 :

Comparison test for (1), (2), (4), (5), (6).

Limit comparison test with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\frac{1}{n^{1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$$

and since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^{1+\frac{1}{n}}}$ diverges.

Pb 2-3 :

2) The sequence $1, -1, 1, -1, 1, \ldots$ is Cesaro summable to $\frac{1}{2}$.

pb 4-3 :

For any N, we have

$$\left|\sum_{n=1}^{N} a_n\right| \leqslant \sum_{n=1}^{N} |a_n| \leqslant \sum_{n=1}^{\infty} |a_n|.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, i.e. $\sum_{n=1}^{\infty} |a_n| < \infty$, passing to the limit as $n \to \infty$ in the above inequality, we get

$$\left|\sum_{n=1}^{\infty} a_n\right| \leqslant \sum_{n=1}^{\infty} |a_n| < \infty.$$

pb 5-3 :

1) We know that

$$|a_n b_n + \cdots + a_m b_m| \leqslant \sqrt{a_n^2 + \cdots + a_m^2}\sqrt{b_n^2 + \cdots + b_m^2}$$

So the Cauchy criterion for $\sum_{n=1}^{\infty} a_n^2$ and

$\sum_{n=1}^{\infty} b_n^2$ implies the Cauchy criterion for the series $\sum_{n=1}^{\infty} a_n b_n$.

2) Taking the convergent series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$, with $b_n = \frac{1}{n^\alpha}$, $\alpha > \frac{1}{2}$ in part (1), we see that $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n \cdot \frac{1}{n^\alpha}$

i.e. $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$, $\alpha > \frac{1}{2}$, is convergent whenever $\sum_{n=1}^{\infty} a_n^2$ is.

pb 6-3:

Since $a_n \geqslant 0$ and $\sum_{n=1}^{\infty} a_n$ converges, we see by Cauchy's criterion that, $\forall \varepsilon > 0$ $\exists N$, $|a_n + \dots + a_m| < \varepsilon$, $\forall m \geqslant n \geqslant N$.

Thus, since (a_n) is decreasing, we get

$$(m-n)\, a_m \leqslant a_n + \dots + a_m < \varepsilon$$

Since $\lim_{m \to \infty} \frac{m}{m-n} = 1$, we see that for $\varepsilon' = 1$ there is N', $\left|\frac{m}{m-n} - 1\right| < 1$, for $m \geqslant N'$

Thus for $m \geqslant n \geqslant \max(N, N')$, we have

$$|m a_m| = \left|\frac{m}{m-n}(m-n) a_m\right| = \left|\left(\frac{m}{m-n} - 1 + 1\right)(m-n) a_m\right|$$

$$\leqslant \left|\frac{m}{m-n} - 1\right| |(m-n) a_m| + |(m-n) a_m|$$

$$< 1 \cdot \varepsilon + \varepsilon = 2\varepsilon$$

Thus $\lim_{m \to \infty} m a_m = 0$ as required.

pb 7-3:-

If $a_n \not\to 0$ as $n \to \infty$, then $\frac{a_n}{1+a_n} \not\to 0$ as well and thus $\sum \frac{a_n}{1+a_n}$ diverges too.

So suppose $a_n \to 0$ as $n \to \infty$. Without loss of generality, we can suppose also that $a_n < 1$ for all n.

Therefore $a_n < 1 \Rightarrow 1 + a_n < 1+1$

$$\Rightarrow \frac{a_n}{2} < \frac{a_n}{1+a_n}, \text{ since } a_n \geqslant 0$$

also $a_n \geqslant 0 \Rightarrow 1 + a_n \geqslant 1$

$$\Rightarrow \frac{1}{1+a_n} \leqslant 1 \Rightarrow \frac{a_n}{1+a_n} \leqslant a_n$$

Thus we infer that

$$\frac{a_n}{2} \leqslant \frac{a_n}{1+a_n} \leqslant a_n$$

By the comparison test, if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.

The converse also holds

pb 8-3:-

1) Consider the root test (for $a > 0$)

$$\lim_{n\to\infty} \sqrt[n]{\frac{a^n n!}{n^n}} = \lim_{n\to\infty} a \frac{\sqrt[n]{n!}}{n} = \frac{a}{e}$$

So we have convergence for $\frac{a}{e} < 1$

and divergence for $\frac{a}{e} > 1$.

For $a = e$ the root and ratio tests fail.
So, observe that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} \quad \text{—} (*)$$

(from a previous exercise).

Thus $\frac{e^n n!}{n^n} > e$, whence $\lim_{n\to\infty} \frac{e^n n!}{n^n} \neq 0$

and the series diverges.

In conclusion $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n} \begin{cases} \text{converges} & a < e \\ \text{diverges} & a > e \\ \text{diverges} & a = e. \end{cases}$

2) We can alternatively use the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}/a^{n+1}(n+1)!}{n^n/a^n n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{a(n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{1}{a}\left(1+\frac{1}{n}\right)^n = \frac{e}{a}$$

So convergence for $\frac{a}{e} < 1$ and divergence for $\frac{a}{e} > 1$.

for $a = e$, we use the second inequality in $(*)$

So $$\frac{n^n}{e^n n!} = \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)^{n+1}}{e^n n!} > \frac{n^n}{(n+1)^{n+1}} =$$

$$= \frac{1}{n+1}\left(\frac{n}{1+n}\right)^n = \frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n} > \frac{1}{2e(n+1)}$$

By the comparison test $\sum_{n=1}^{\infty} \frac{n^n}{e^n n!}$ diverges since $\sum \frac{1}{n+1}$ diverges.

Exercises on chapter IV

Exercise 1 :-

Find the following limits :-

1) $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

2) $\lim\limits_{h \to 0} \dfrac{\sqrt{a+h} - \sqrt{a}}{h}$

3) $\lim\limits_{x \to 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$

4) $\lim\limits_{x \to 0} \dfrac{\sin^2 2x}{x}$

5) $\lim\limits_{x \to 0} \dfrac{\tan^2 x + 2x}{x + x^2}$

6) $\lim\limits_{x \to 0} \dfrac{x \sin x}{1 - \cos x}$

7) $\lim\limits_{x \to 1} \dfrac{\sin(x^2 - 1)}{x - 1}$

8) $\lim\limits_{x \to \infty} \dfrac{x \sin x}{x^2 + 5}$

9) $\lim\limits_{x \to 0} \dfrac{x^2 (1 + \sin^2 x)}{(x + \sin x)^2}$

10) $\lim\limits_{x \to \infty} \dfrac{x^3 + 4x - 7}{7x^2 - x + 1}$

11) $\lim\limits_{x \to \infty} \sqrt{x^2 + 2x} - x$

12) $\lim\limits_{x \to \infty} x\left(\sqrt{x+2} - \sqrt{x}\right)$

Exercise 2 :-

Show that $\lim\limits_{x \to 0^+} f\left(\dfrac{1}{x}\right) = \lim\limits_{x \to \infty} f(x)$.

Exercise 3 :-

Using the definition, show that

$$\lim_{x \longrightarrow 0} \frac{x}{1+\sin^2 x} = 0$$

Exercise 4 :-

Show that $\lim_{x \longrightarrow a} f(x-a) = \lim_{x \longrightarrow 0} f(x)$.

Exercise 5 :-

Using the definition, show that

$$\lim_{x \longrightarrow a} x+b = a+b.$$

Limits :-

Exo 2 :-

If $\ell = \lim_{x \to \infty} f(x)$, then

$\forall \varepsilon > 0$, $\exists M > 0$ such that for $x > M$ we have
$$|f(x) - \ell| < \varepsilon.$$

Now, if $0 < x < \frac{1}{M}$, then $\frac{1}{x} > M$

So $|f(\frac{1}{x}) - \ell| < \varepsilon$

i.e., $\forall \varepsilon > 0$, $\exists \delta = \frac{1}{M}$ such that $|f(\frac{1}{x}) - \ell| < \varepsilon$ for $0 < x < \delta = \frac{1}{M}$.

i.e. $\lim_{x \to 0^+} f(\frac{1}{x}) = \ell$.

Since the reciprocal of this implication is true, we obtain the desired equivalence.

Exo 3 :- ✓

Since $\left|\frac{x}{1 + \sin^2 x}\right| \leqslant |x|$, just take $\delta = \varepsilon$.

Exo 4 :-

$\lim_{x \to a} f(x-a) = \ell \iff$

$\forall \varepsilon > 0$, $\exists \delta > 0$, $|x-a| < \delta \implies |f(x-a) - \ell| < \varepsilon$

put $y = x - a$, then

$\forall \varepsilon > 0$, $\exists \delta > 0$, $|y| < \delta \implies |f(y) - \ell| < \varepsilon$

$$\iff \lim_{y \to 0} f(y) = \ell$$

i.e. $\lim_{x \to 0} f(x) = \ell$

Exercises for chapter V :

Exercise 1 :
Find c such that f is continuous

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & , \; x \neq 2 \\ c & , \; x = 2. \end{cases}$$

Exercise 2 :
Suppose that f satisfies $f(x+y) = f(x) + f(y)$
and that f is continuous at 0.
Show that f is continuous at any $a \in \mathbb{R}$.

Exercise 3 :
Show that the equation $x\,2^x = 1$ has a solution in $(0, 1)$.

Exercise 4 :
Show that the equation $\cos x = x$ has a solution in $(0, \frac{\pi}{2})$.

Exercise 5 :
Let f be continuous on $[0, 1]$, and suppose $f(0) = f(1)$. Show the existence of a point $c \in [0, \frac{1}{2}]$ where $f(c) = f(c + \frac{1}{2})$.
Hint : use $g(x) = f(x) - f(x + \frac{1}{2})$.

Exercise 6 :
Show that $f(x) = \sin x$ is continuous at any $a \in \mathbb{R}$.

Continuity :-

Exo 2 :-

Note that $f(x+0) = f(x) + f(0)$, so $f(0) = 0$.

Now, $\lim_{h \to 0} f(a+h) - f(a) = \lim_{h \to 0} \cancel{f(a)} + f(h) - \cancel{f(a)}$

$$= \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} f(h) - f(0) = 0$$

(as f continuous at 0)

So $\lim_{h \to 0} f(a+h) = f(a)$, i.e. f continuous.

Exo 3 :-

Set $f(x) = x\,2^x$, which is continuous on $\mathbb{R}$ and in particular on $[0,1]$.

Now $f(0) = 0$, $f(1) = 1 \cdot 2^1 = 2$.

Thus $f(0) = 0 < 1 < 2 = f(1)$; by the intermediate value theorem, there is a $c \in (0,1)$ such that

$$f(c) = c\,2^c = 1,$$

Thus the equation $x\,2^x = 1$ admits a solution $c \in (0,1)$.

Exo 4 :-

Set $f(x) = \cos x - x$, continuous on $[0, \frac{\pi}{2}]$.

$f(0) = \cos 0 - 0 = 1$, $f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2}$

Thus $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0 < 1 = \cos 0$. By the

intermediate value theorem, there is a $c \in (0, \frac{\pi}{2})$
such that $f(c) = 0$, i.e. $\cos c = c$.
Thus the equation $\cos x = x$ admits a solution in $[0, \frac{\pi}{2}]$.

Exercise 5 :-

Set $g(x) = f(x) - f(x + \frac{1}{2})$, on $[0, \frac{1}{2}]$.
g is continuous on $[0, \frac{1}{2}]$, and

$$g(0) = f(0) - f(\tfrac{1}{2}) \;,\; g(\tfrac{1}{2}) = f(\tfrac{1}{2}) - f(1) = f(\tfrac{1}{2}) - f(0) = -g(0).$$

Thus, 0 lies between $g(0)$ and $g(\frac{1}{2})$. By the intermediate value theorem there is a $c \in [0, \frac{1}{2}]$ such that $g(c) = 0$.
i.e. $f(c) = f(c + \frac{1}{2})$.

Exercise 6 :-

Given $\varepsilon > 0$, there is $0 < \delta < \pi$ such that

$$|\sin x - \sin a| = \left|2 \cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right)\right|$$

$$\leq 2 \left|\sin\left(\frac{x-a}{2}\right)\right| \leq 2 \left|\frac{x-a}{2}\right| < \delta = \varepsilon$$

(valid for $\left|\frac{x-a}{2}\right| < \frac{\pi}{2}$), ($|\sin x| \leq |x|$ for x small).

i.e. $\lim\limits_{x \to a} \sin x = \sin a$.

Exercises for chapter VI :

Exercise 1 :

Find $f'(x)$ in each case :

1) $f(x) = \sin\left(\frac{\cos x}{x}\right)$, 2) $f(x) = (\sin x)^{\frac{\cos x}{x}}$.

Exercise 2 :

Find the value of λ that makes f differentiable at $x = 0$, where

$$f(x) = \begin{cases} \sin x , & x > 0 \\ 0 , & x = 0 \\ x^2 + \lambda x , & x < 0 \end{cases}$$

Exercise 3 :

1) Find the extrema of $f(x) = x^3 - x$ on $[-1, 2]$.

2) Same question for $g(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Exercise 4 :

Sketch the graph of the functions :

1) $f(x) = x^4 - 2x^2$

2) $g(x) = \dfrac{x^2 - 2x + 2}{x - 1}$

3) $h(x) = x + \dfrac{1}{x}$.

Exercise 5 :

Show that of all rectangles with given perimeter, the square has the greatest area.

Exercise 6 :

Show that the sum of a positive number and its reciprocal is at least 2.

Exercise 7:
Find among all right circular cylinders of fixed volume V, the one with smallest surface area, (counting the areas of top and bottom).

Exercise 8:
Show that if $\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$
then $a_0 + a_1 x + \cdots + a_n x^n = 0$
for some x in $[0, 1]$.

Exercise 9:
Find $f'(0)$ if $f(x) = \begin{cases} \frac{g(x)}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

and $g(0) = g'(0) = 0$ and $g''(0) = 17$.

Exercise 10:
Use Taylor's theorem with $n = 2$ to obtain a suitable approximation of $\sqrt{0.9}$.

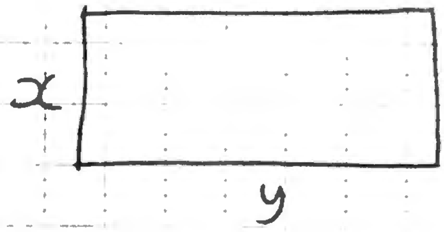
Exercise 11:
1) Show that for every $a > 0$, the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$.

2) Show that $g(x) = e^{-x}$ is uniformly continuous on $[0, \infty)$

3) Show that $h(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

Derivatives :

Exo 5 :



$$\text{perimeter} = 2x + 2y = L$$

Area : $A(x,y) = xy$

$$2x + 2y = L \Rightarrow y = \frac{1}{2}(L - 2x)$$

$$\Rightarrow A(x) = x \cdot \frac{1}{2}(L - 2x) = - x^2 + \frac{L}{2}x$$

$$A'(x) = -2x + \frac{L}{2} = 0 \Rightarrow x = \frac{L}{4}$$

$$A''(x) = -2 < 0 \Rightarrow A\left(\frac{L}{4}\right) \text{ local max.}$$

Now, $x = \frac{L}{4}$ and $y = \frac{1}{2}(L - 2x)$, we get

$$y = \frac{1}{2}\left(L - 2\frac{L}{4}\right) = \frac{L}{4}. \text{ Thus } x = y$$

$\Rightarrow$ this rectangle should be a square.

Exo 6 :

Suppose $x > 0$, then put $f(x) = x + \frac{1}{x}$

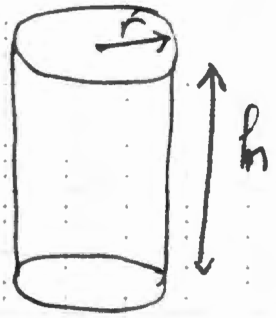
$$f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1, \text{ but } x > 0$$

Thus $x = 1$ is a critical point.

$f''(x) = +\frac{2}{x^3}$, $f''(1) = +2 > 0 \Rightarrow f(1) = 2$ is a local minimum, thus $x + \frac{1}{x} \geqslant 2$.

Exo 7 :

Volume $= \pi r^2 h = \alpha$ (fixed)



Surface area $= 2\pi r h + 2\pi r^2$

$$S(r,h) = 2\pi r h + 2\pi r^2$$

but $\pi r^2 h = \alpha \Rightarrow h = \dfrac{\alpha}{\pi r^2}$

$$\Rightarrow S(r) = 2\cancel{\pi}\cancel{r}\cdot\frac{\alpha}{\cancel{\pi} r^{\cancel{2}}} + 2\pi r^2$$

$$\Rightarrow S(r) = \frac{2\alpha}{r} + 2\pi r^2$$

$$A'(r) = -\frac{2\alpha}{r^2} + 4\pi r = 0$$

$$\Rightarrow r^3 = \frac{\alpha}{2\pi} \Rightarrow \boxed{r = \sqrt[3]{\frac{\alpha}{2\pi}}}$$

But $h = \dfrac{\alpha}{\pi r^2} \Rightarrow \boxed{h = \dfrac{\alpha}{\pi\left(\sqrt[3]{\frac{\alpha}{2\pi}}\right)^2} = \dfrac{\alpha}{\pi}\left(\dfrac{2\pi}{\alpha}\right)^{\frac{2}{3}}}$

So $A''(r) = \dfrac{4\alpha}{r^3} + 4\pi$ so $A''(r)\Big/_{r=\sqrt[3]{\frac{\alpha}{2\pi}}} > 0$

$\Rightarrow A\left(\sqrt[3]{\frac{\alpha}{2\pi}}\right)$ is a local minimum.

Thus the desired cylinder is the one whose radius is $r = \sqrt[3]{\frac{\alpha}{2\pi}}$ and height is

$$h = \frac{\alpha}{\pi}\left(\frac{2\pi}{\alpha}\right)^{2/3}.$$

Exo 8 :-

Consider the function

$$f(x) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots + \frac{a_n}{n+1} x^{n+1}$$

f is continuous in $[0, 1]$, differentiable on $(0, 1)$, and

$$f(0) = 0 \quad , \quad f(1) = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1}$$

so, if $a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1} = 0$, then $f(0) = f(1)$.

Rolle's theorem is applicable and gives

$\exists\, c \in (0, 1)$ such that $f'(c) = 0$

$$f'(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\Rightarrow f'(c) = a_0 + a_1 c + a_2 c^2 + \dots + a_n c^n = 0$$

i.e., there is some $x \in [0, 1]$ such that

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0.$$

Exo 9 :-

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{g(x)}{x} - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{g(x)}{x^2} = \lim_{x \to 0} \frac{g'(x)}{2x} = \lim_{x \to 0} \frac{\overset{17}{g''(0)}}{2}$$

Thus $f'(0) = \frac{17}{2}$.

Exo 10:

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$.

Near 1, we have

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(c)}{3!}(x-1)^3.$$

for some c between x and 1.

Thus for $x = 0.9$

$$f(0.9) \approx 1 + \frac{1}{2}(0.9-1) + \frac{1}{8}(0.9-1)^2$$

$$\approx 1 - 0.05 - 0.00125 = 0.94875$$

So the suitable approximation by Taylor's theorem for $n = 2$ is

$$\sqrt{0.9} \approx \underline{\underline{0.948}}75 .$$

Note that by calculator $\sqrt{0.9} \approx \underline{\underline{0.948}}6832980505\ldots$

Solution of Exercise 11:

1) For $x, y \geqslant a$, we have

$$|f(x) - f(y)| = \frac{|x-y|}{|xy|} \leqslant \frac{1}{a^2}|x-y|$$

Then we can apply the definition of uniform continuity, (complete the solution).

2) $g'(x) = -e^{-x} \Rightarrow |g'(x)| = e^{-x} \leqslant 1$, for $x \geqslant 0$

Thus g has a bounded derivative for $x \geqslant 0$ whence it is uniformly continuous there.

3) The mean value theorem applied for $h(x) = \sin x$ on $[x, y]$, $x, y \in \mathbb{R}$ arbitrary gives

$|\sin x - \sin y| \leqslant |x - y|$, then apply the definition.

problems for chapter VII :-

problem 1 - 7 :-

let $0 < a < b$ and consider the function :-

$$f(x) = \begin{cases} 0 , & x \in [a, b] \cap \mathbb{Q} \\ x , & x \in [a, b] \cap \mathbb{Q}^c . \end{cases}$$

1) Find Riemann's upper and lower integrals $\mathcal{U}$ and $\mathcal{L}$.

2) Verify whether $f \in R(a, b)$.

Solution :-

Let P be a partition of $[a, b]$ such that

$$P = \{ x_1 = a < x_2 < \dots < x_n = b \}$$

then $\sup \{ f(x), x_i < x < x_{i+1} \} = x_{i+1}$

and $\inf \{ f(x), x_i < x < x_{i+1} \} = 0$

this is due to the density of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$.

Hence $L(P, f) = 0$ and

$$U(P, f) = x_2 (x_2 - x_1) + \dots + x_n (x_n - x_{n-1})$$

$$= U(P, h).$$

Here h is the function $h : [a, b] \longrightarrow \mathbb{R}$ defined by $h(x) = x$.

Since h is continuous, it is integrable

and $\quad \mathcal{U}(h) = \inf \mathcal{U}(P, h) = \frac{b^2 - a^2}{2}.$

where the infimum is taken over all partitions of $[a, b]$.

Thus we obtain :-

$L(f) = \sup L(P, f) = 0$, $U(f) = \inf U(P, f) = \frac{b^2 - a^2}{2}$

Now, since $L(f) = 0 < U(f) = \frac{b^2 - a^2}{4}$, we see that $f \notin R(a, b)$.

problem 2_7 :-

1) show that if $f \in R(0,1)$ then

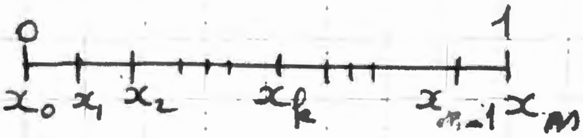
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx .$$

g^{x} ← 2) use (1) to calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3 + k^3}$$

solution :-

If $f \in R(0, 1)$, let us choose a uniform partition P_n such that



then

$$x_k = \frac{k}{n} , \quad x_{k+1} - x_k = \frac{1}{n} , \quad w_{k+1} = x_{k+1} = \frac{k+1}{n}$$

$$\Rightarrow S(f, P_n) = \sum_{k=1}^{n} f(w_k)\,\Delta x_k = \sum_{k=0}^{n-1} f\left(\frac{k+1}{n}\right)\cdot\frac{1}{n}$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$, (put $i = k+1$ and then return back to k).

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

2) To calculate this limit, we write $\frac{k^2}{n^3+k^3}$ in the form $\frac{1}{n} f\left(\frac{k}{n}\right)$. So

$$\frac{k^2}{n^3+k^3} = \frac{1}{n^3} \frac{k^2}{1+(k/n)^3} = \frac{1}{n} \frac{\left(\frac{k}{n}\right)^2}{1+(k/n)^3} = \frac{1}{n} f\left(\frac{k}{n}\right)$$

where $f(x) = \dfrac{x^2}{1+x^3}$.

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3+k^3} = \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$= \frac{1}{3} \ln(1+x^3)\Big/_0^1 = \frac{1}{3}(\ln 2 - \ln 1)$$

$$= \frac{\ln 2}{3}.$$

problem 3-7:-

let f be a continuous function on $[a, b]$, and define the function F as follows:-

$$f :- [a, b] \longrightarrow \mathbb{R}$$

$$x \longrightarrow F(x) = \int_a^x f(t)\,dt.$$

using the defintion, show that

$$F'(c) = f(c), \ \forall\, c \in [a, b].$$

Solution:-

f is continuous at any $c \in [a, b]$. So

$\forall \varepsilon>0, \exists \delta>0, |x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon.$

Now

$$\left|\frac{F(x)-F(c)}{x-c}-f(c)\right| = \left|\frac{1}{x-c}\int_c^x (f(t)-f(c))\,dt\right|$$

$$\leqslant \frac{1}{|x-c|}\left|\int_c^x \varepsilon\,dt\right| = \varepsilon$$

i.e. at any $c\in[a,b]$ one has

$$\lim_{x\to c}\frac{F(x)-F(c)}{x-c} = f(c) \Rightarrow F'(c)=f(c).$$

problem 4-7:-

Use Riemann's sums to calculate the limits

1) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{4n^2+k^2}$

2) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\tan^{-1}\left(\frac{n}{n^2+k^2}\right)$

3) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\tan\left(\frac{k\pi}{4n+4}\right).$

Solution:-

1) Let $f(x)=\dfrac{1}{4+x^2}$ on $[0,1]$. Then we get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{4n^2+k^2} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$

2) we have $\forall x>0,\ \tan^{-1}x > x-\dfrac{x^3}{3}.$

Thus

$$\frac{n}{n^2+k^2}-\frac{1}{3}\left(\frac{n}{n^2+k^2}\right)^3 \leqslant \tan^{-1}\left(\frac{n}{n^2+k^2}\right)\leqslant \frac{n}{n^2+k^2}$$

on the other hand, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \int_0^1\frac{dx}{1+x^2} = \frac{\pi}{4}$$

Also $\sum_{k=1}^{n}\left(\frac{n}{n^2+k^2}\right)^3 \leqslant \sum_{k=1}^{n}\left(\frac{n}{n^2}\right)^3 = \sum_{k=1}^{n}\frac{1}{n^3} = \frac{1}{n^2}$.

$$\Longrightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n}{n^2+k^2}\right)^3 = 0$$

Thus $\lim_{n\to\infty}\sum_{k=1}^{n}\tan^{-1}\left(\frac{n}{n^2+k^2}\right) = \frac{\pi}{4}$.

3) using the function $f(x) = \frac{\tan x}{x}$, we get

$$\sum_{k=1}^{n}\frac{1}{k}\tan\left(\frac{k\pi}{4n+4}\right) = \sum_{k=1}^{n}\frac{\pi}{4n+4}f\left(\frac{k\pi}{4n+4}\right)$$

and we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{4n+4}f\left(\frac{k\pi}{4n+4}\right) = \int_0^{\frac{\pi}{4}} f(x)\,dx.$$

Thus, we get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\tan\left(\frac{k\pi}{4n+4}\right) = \int_0^{\frac{\pi}{4}}\frac{\tan x}{x}\,dx.$$

observe that the function $\frac{\tan x}{x}$ can be extended at $x=0$ on $[0, \frac{\pi}{4}]$ by setting $f(0)=1$, and therefore $\int_0^{\pi/4}\frac{\tan x}{x}\,dx$ exists (Converges).

problem 5-7:

Calculate the following limits:-

1) $\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+x}\,dx$

2) $\lim_{n\to\infty}\int_0^1 n x(1-x^2)^n\,dx$

3) $\lim_{n\to\infty}\int_1^2 \frac{\sin(nx)}{x}\,dx$

Solution:

1) $$\int_0^1 \frac{x^n}{1+x}\,dx = \int_0^\delta \frac{x^n}{1+x}\,dx + \int_\delta^1 \frac{x^n}{1+x}\,dx$$

we have

$$0 \leqslant \int_0^\delta \frac{x^n}{1+x}\,dx \leqslant \int_0^\delta x^n\,dx \leqslant \delta^n \int_0^\delta dx = \delta^{n+1}$$

$$\Longrightarrow \lim_{n\to\infty} \int_0^\delta \frac{x^n}{1+x}\,dx = 0.$$

Fix $\varepsilon \in (0,1)$ and put $\delta = 1 - \frac{\varepsilon}{2}$, then $\delta \in (0,1)$ and there is an $N \geqslant 1$ such that

$$0 \leqslant \int_0^\delta \frac{x^n}{1+x}\,dx < \frac{\varepsilon}{2}, \quad \forall\, n \geqslant N$$

on the other hand, we have

$$0 \leqslant \int_\delta^1 \frac{x^n}{1+x}\,dx \leqslant \int_\delta^1 dx = 1 - \delta = \frac{\varepsilon}{2}.$$

$$\Longrightarrow 0 \leqslant \int_0^1 \frac{x^n}{1+x}\,dx < \varepsilon, \quad \forall\, n \geqslant N$$

i.e. $$\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\,dx = 0.$$

2) Direct calculation leads to

$$\lim_{n\to\infty} \int_0^1 n x (1-x^2)^n\,dx = \frac{1}{2}.$$

3) Integration by parts gives:

$$\int_1^2 \frac{\sin(nx)}{x}\,dx = \left[-\frac{\cos(nx)}{nx}\right]_1^2 + \int_1^2 \frac{\cos(nx)}{nx}\,dx$$

$$= \frac{\cos n}{n} - \frac{\cos 2n}{2n} + \frac{1}{n}\int_1^2 \frac{\cos nx}{x}\,dx$$

Since $\left|\frac{\cos 2n}{2n}\right| \leqslant \frac{1}{2n}$, $\left|\frac{\cos n}{n}\right| \leqslant \frac{1}{n}$ and

$\left|\int_1^2 \frac{\cos nx}{n}\,dx\right| \leqslant \int_1^2 \left|\frac{\cos nx}{x}\right| dx \leqslant \int_1^2 \frac{dx}{x} = \ln 2$, whence

$$\lim_{n\to\infty} \int_1^2 \frac{\sin(nx)}{n}\,dx = 0.$$

problem 6-7:

Test the Convergence of the following improper integral

1) $\int_1^{\infty} \frac{dx}{\sqrt{1+x^3}}$

2) $\int_1^{\infty} \frac{\sin x}{x} dx$. Also, is it absolutely convergent?

3) $\int_0^{\infty} \frac{x}{1+x^2 \sin^2 x} dx$

Solution:

1) $\frac{1}{\sqrt{1+x^3}} \leqslant \frac{1}{x^{3/2}}, \ \forall \ x \geqslant 1.$

Since the "Riemann" improper integral $\int_1^{\infty} \frac{dx}{x^{3/2}}$ is convergent (as $3/2 > 1$), we see that

$$\int_1^{\infty} \frac{dx}{\sqrt{1+x^3}} \text{ is convergent.}$$

2) Integration by parts:

$$\int_1^{A} \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_1^A - \int_1^A \frac{\cos x}{x^2} dx, \ \forall \ A > 1.$$

Since $\lim_{A \to \infty} \left[-\frac{\cos x}{x}\right]_1^A = \frac{\cos 1}{1} = \cos 1$ and the improper integral $\int_1^{\infty} \frac{\cos x}{x^2} dx$ is absolutely convergent (since $\left|\frac{\cos x}{x^2}\right| \leqslant \frac{1}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} < \infty$), we infer that $\int_1^{\infty} \frac{\sin x}{x} dx$ is convergent.

Let us use contradiction to show that it is not absolutely convergent. So suppose $\int_1^{\infty} \left|\frac{\sin x}{x^2}\right| dx$ is convergent. We know that

$$\int_1^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx$$

The latter series is then convergent.

But $\displaystyle\int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \int_0^\pi \frac{|\sin(x+n\pi)|}{x+n\pi} dx$

$$= \int_0^\pi \frac{|(-1)^n \sin x|}{x+n\pi} dx = \int_0^\pi \frac{\sin x}{x+n\pi} dx$$

Since

$\displaystyle\int_0^\pi \frac{\sin x}{x+n\pi} dx \geqslant \int_0^\pi \frac{\sin x}{\pi+n\pi} = \frac{2}{(n+1)\pi}$, we see

that $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(n+1)\pi}$ is convergent, which is a contradiction.

3) $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^2\sin^2 x}$ convergent $\Longleftrightarrow \displaystyle\int_1^\infty \frac{x\,dx}{1+x^2\sin^2 x}$ convergent.

Since

$\displaystyle\frac{x}{1+x^2\sin^2 x} \geqslant \frac{x}{1+x^2} \geqslant \frac{1}{2x}$, $\forall\, x \geqslant 1$, we see

that $\displaystyle\int_1^\infty \frac{x\,dx}{1+x^2\sin^2 x}$ is divergent, because

$\displaystyle\int_1^\infty \frac{dx}{2x}$ is divergent.

Hence, the integral $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^2\sin^2 x}$ is divergent.

problem 7-7:-

Discuss the convergence of the improper integral

$$I_a = \int_2^\infty \frac{dx}{x(\ln x)^a}, \quad a > 0.$$

solution :-
put $y = f(x)$ and differentiate both sides to obtain

$$2yy' = y\cos^2 x - 2\cos x \sin x \cdot y^2 \sec^2 x$$

$$\Longrightarrow \quad y' + \tan x \cdot y = \cos^2 x$$

whose general solution (see the hint) is

$$y = \frac{1}{2}\sin x \cos x + C(\cos x).$$

But if $x = 0$, from the equation

$$f^2(x) = \cos^2 x \int_0^x f(t)\,dt$$

we see that $f(0) = y/_{x=0} = 0$, whence

$$C = 0 \quad \Longrightarrow f(x) = \frac{1}{2}\sin x \cos x.$$

problem 9-7 :-
Calculate the limit $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2}$.

solution :
Same method as problem 2-7.

α problem 10-7 :-
Test the convergence of the improper integral

$$I = \int_0^\infty \frac{4x}{1+x^6}\,dx.$$

problems for chapter VIII:

problem 1-8:

Study the convergence of the sequences:

1) $f_n(x) = \dfrac{1}{1+(nx-1)^2}$, $D = [0,1]$.

2) $f_n(x) = n\,x^n(1-x)$, $D = [0,1]$.

3) $f_n(x) = \tan^{-1}\left(\dfrac{2x}{x^2+n^3}\right)$, $D = \mathbb{R}$.

Solution:

1) The limit is $f(x) = \begin{cases} 0 & , x \in (0,1] \\ \frac{1}{2} & , x = 0 . \end{cases}$

f_n is continuous on $[0,1]$ $\forall n \in \mathbb{N}$, while f is not continuous on $[0,1]$. So the convergence is not uniform.

2) The pointwise limit is 0, i.e. $\lim\limits_{n\to\infty} f_n(x) = 0$.

For any $n \geqslant 2$, we have

$$f_n'(x) = n\,x^{n-1}(n-(n+1)x)$$

So $f_n'(x) = 0 \iff x = 0$ or $x = \dfrac{n}{n+1}$.

$\sup\{|f_n(x) - 0|, x \in [0,1]\} = f_n\left(\frac{n}{n+1}\right)$.

But $\lim\limits_{n\to\infty} f_n\left(\dfrac{n}{n+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{n+1} = \dfrac{1}{e} \neq 0$

So the convergence is not uniform.

3) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1}\left(\frac{2x}{x^2+n^3}\right) = 0$

To find max, we differentiate

$$f_n'(x) = \frac{2n^3 - 2x^2}{(x^2+n^3)^2 + 4x^2}$$

$f_n'(x) = 0 \iff x = \pm n\sqrt{n}$, whence

$\sup\{|f_n(x) - 0|, x \in \mathbb{R}\} = f_n(n\sqrt{n}) = \tan^{-1}\left(\frac{1}{n\sqrt{n}}\right)$

Since $\lim_{n\to\infty} \sup\{|f_n(x) - 0|, x \in \mathbb{R}\} = 0$,

we see that $\{f_n\}$ is uniformly convergent

to 0 on $\mathbb{R}$.

problem 2-8:-

Let $\{f_n\}$ be a sequence of functions

$f_n : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f_n(x) = \frac{nx}{\sqrt{1+n^2x^2}}.$$

Find the pointwise limit of (f_n).

Is it uniformly convergent?

Answer:- (develop the solution)

The limit is $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

The convergence is not uniform.

problem 3_8 :-

Let $f_n(x) = \dfrac{nx + \sin(nx^2)}{n}$.

show that the following limit exists and find it

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx.$$

Answer :- (Develop the solution)

$$f_n \xrightarrow{u} f(x) = x \quad \text{on } [0, 1].$$

So $\displaystyle \int_0^1 f_n(x)\, dx \xrightarrow[\text{as } n \to \infty]{} \int_0^1 f(x)\, dx = \frac{1}{2}$

problem 4 – 8 :-

Let $f_n(x) = \dfrac{n + \sin nx}{3n + \sin^2 nx}$, $x \in \mathbb{R}$.

1) show that (f_n) is uniformly convergent on $\mathbb{R}$.

2) Calculate $\displaystyle \lim_{n \to \infty} \int_0^{\pi} f_n(x)\, dx$.

Solution :- (Develop the solution)

1) By the Sandwich theorem we have

$$f_n \xrightarrow{u} f(x) = \frac{1}{3}.$$

2) $\displaystyle \lim_{n \to \infty} \int_0^{\pi} f_n(x)\, dx = \int_0^{\pi} \frac{1}{3}\, dx = \frac{\pi}{3}.$

problem 5_8 :
Let $f_n, g_n : [0,1] \longrightarrow \mathbb{R}$
be such that

$$f_n(x) = \frac{nx^2}{1+n^2x^2}, \quad g_n(x) = \frac{n^2 x}{1+n^2x^2}.$$

1) Find the pointwise limits of (f_n) and (g_n).

2) Study their uniform convergence.

Answer :

$$g_n(x) \longrightarrow g(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

Since g is not continuous, while g_n is continuous $\forall n$, the convergence is not uniform.

$f_n \longrightarrow f(x) = 0$, and $\forall \varepsilon > 0$ there is $N \geq \frac{1}{\varepsilon}$ such that

$$n \geq N \Rightarrow |f_n(x)| = \frac{1}{n}\left(\frac{nx^2}{\frac{1}{n} + nx^2}\right) \leq \frac{1}{n} < \varepsilon$$

$\forall x \in [0,1]$.

Thus $f_n \xrightarrow{u} f$.

problem 6-8 :-

Show that $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$ is continuous on $\mathbb{R}$.

Solution :-

put $M_n = \left(\frac{t^n}{n!}\right)^2$, $\forall n \geq 1$, $t > 0$

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \left(\frac{t^{n+1}}{(n+1)!}\right)^2 \left(\frac{n!}{t^n}\right)^2$$

$$= \lim_{n\to\infty} \left(\frac{t}{n+1}\right)^2 = 0 < 1.$$

the M-test shows that $\sum_{n=0}^{\infty} M_n$ is Convergent.

Now, $\forall$ $x \in [-t, t]$, we have

$$|f_n(x)| = \left|\left(\frac{x^n}{n!}\right)^2\right| \leq M_n$$

By weirstrass test $\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^n$ is uniformly Convergent on $[-t, t]$ for any $t > 0$

Since $\left(\frac{x^n}{n!}\right)^2$ is continuous for any $n \in \mathbb{N}$

we see that $f(x)$ is also Continuous.

problem 7-8 :-

Let $f_n : [1, 2] \longrightarrow \mathbb{R}$, such that

$$f_n(x) = \frac{x}{(1+x)^n}$$

1) Show that $\sum_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in [1,2]$.

2) Show that this convergence is uniform.

3) Verify the identity

$$\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)\, dx.$$

Solution:-

1) If $|1+x| > 1$, (i.e. $\left|\frac{1}{1+x}\right| < 1$), we have

$$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n} = x \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{x}{1+x} \frac{1}{1-\frac{1}{1+x}} = 1.$$

In particular $\sum_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in [1,2]$.

2) Since $1 \leq x \leq 2$, we have $1+x \geqslant 2$

$$\Longrightarrow \frac{1}{(1+x)^n} \leq \frac{1}{2^n} \Longrightarrow \frac{x}{(1+x)^n} \leq \frac{x}{2^n} \leq \frac{2}{2^n}$$

Take $M_n = \frac{1}{2^{n-1}}$, so $|f_n(x)| = \left|\frac{x}{(1+x)^n}\right| \leq \frac{1}{2^{n-1}}$

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent, then $\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}$ is uniformly convergent by Weiristrass.

3) The uniform convergence guarantees the introduction of integral under the infinite sum, so

$$\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)\, dx.$$

problem 8-8:

Show that $\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n(n+2)} dx = 0.$

solution:

$\forall n \in \mathbb{N}$, we have

$$\left| \frac{\cos 2nx}{n(n+2)} \right| \leq \frac{1}{n(n+2)} \quad , \forall x \in [0, \pi].$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent, by weirstrass test $\sum_{n=1}^{\infty} \frac{\cos 2nx}{n(n+2)}$ is uniformly convergent.

Now, $\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\cos 2nx}{n(n+2)} \right) dx =$

$$= \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\cos 2nx}{n(n+2)} dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \left[\frac{\sin 2nx}{2n} \right]_0^{\pi}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n^2(n+2)} (\sin 2n\pi - \sin 0)$$

$$= 0 \quad , \quad (\text{as } \sin 2n\pi = \sin 0 = 0)$$

problem 9-8:

1) Find the function $f(x)$ which is given by $\sum_{n=1}^{\infty} n^2 x^n \; , \; |x| < 1.$

2) Calculate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=2}^{\infty} \frac{n^2}{2^n}$

Solution :-

For $|x|<1$, we have $\frac{1}{1-x} = 1+x+x^2+\cdots$

Differentiating twice, we get

$$\frac{1}{(1-x)^3} = 1+3x+6x^2+10x^3+\cdots$$

Multiplying by x, and then by x^2 we get

$$\frac{x}{(1-x)^3} = x+3x^2+6x^3+10x^4+\cdots$$

$$\frac{x^2}{(1-x)^3} = x^2+3x^3+6x^4+10x^5+\cdots$$

Adding the terms, we get

$$\frac{x+x^2}{(1-x)^3} = x+4x^2+9x^3+16x^4+\cdots$$
$$= \sum_{n=1}^{\infty} n^2 x^4 . \quad \Rightarrow f(x) = \frac{x+x^2}{(1-x)^3}.$$

2) $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = f\left(\frac{1}{2}\right) = 6.$

$$\sum_{n=2}^{\infty} \frac{n^2}{2^n} = f\left(\frac{1}{2}\right) - \frac{1}{2} = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} - \frac{1}{2} = \frac{11}{2}.$$

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problem 10-8 :-

Find the power series of $\log\left(\frac{1+x}{1-x}\right)$, $x\in(-1,1)$.

Solution :-

Since $[-x, x] \subset (-1, 1)$, we have $|t^n| \leq x^n < 1$, whence the power series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ is uniformly convergent on $[-x, x]$ for $x \in (-1,1)$, by weirstrass test.

Now $\log\left(\frac{1+x}{1-x}\right) = \int_{-x}^{x} \frac{dt}{1-t} = \int_{-x}^{x} \sum_{n=0}^{\infty} t^n \, dt$

$$= \sum_{n=0}^{\infty} \int_{-x}^{x} t^n \, dt = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(x^{n+1} - (-x)^{n+1}\right)$$

$$= \sum_{n \text{ odd}}^{\infty} \frac{2x^n}{n} = \sum_{n=1}^{\infty} \frac{2x^{2n-1}}{2n-1}$$

Exercise:

a) Find the radius of convergence and the intervals of convergence of each the power series:

1) $\sum_{n=1}^{\infty} n x^n$

2) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} (x-5)^n$

3) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!} x^n$.

4) $\sum_{n=0}^{\infty} \frac{n+1}{10^n} (x-4)^n$

5) $\sum_{n=1}^{\infty} \frac{(-2x)^n}{n^4}$.

b) write $\frac{\ln(1+x)}{x}$ in the form of a power series.