INTEGRAL CALCULUS (MATH 106)

Dr. Borhen Halouani

king saud university

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Integration By Parts

It is used to solve integration of a product of two functions using the formula:

$$\int u \, dv = uv - \int v \, du$$

1. $\int xe^x dx$
   $u = x \quad dv = e^x \, dx$
   $du = dx \quad v = e^x$
   $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$
2. $\int_0^\pi x \sin x \, dx$
   $u = x \quad dv = \sin x \, dx$
   $du = dx \quad v = -\cos x$
   $\int_0^\pi x \sin x \, dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi$
   $[(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = \pi$

Notes:

1. $\int xe^x \; dx = (x-1)e^x + c$
   $\int x^2e^x \; dx = (x^2 - 2x + 2)e^x + c$
   $\int x^3e^x \; dx = (x^3 - 3x^2 + 6x - 6)e^x + c$
2. $\int x \cos x \; dx = x \sin x + \cos x + c$
   $\int x^2 \cos x \; dx = (x^2 - 2) \sin x + 2x \cos x + c$
   $\int x^3 \cos x \; dx = (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c$
   $\int x^4 \cos x \; dx = (x^4 - 12x^2 + 24) \sin x + (4x^3 - 24x) \cos x + c$
3. $\int x \sin x \; dx = -x \cos x + \sin x + c$
   $\int x^2 \sin x \; dx = (-x^2 + 2) \cos x + 2x \sin x + c$
   $\int x^3 \sin x \; dx = (-x^3 + 6x) \cos x + (3x^2 - 6) \sin x + c$
   $\int x^4 \sin x \; dx = (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + c$

Integrals Involving Trigonometric Functions

First : Integrals of the forms

$$\int \sin ax \cos bx \; dx, \quad \int \sin ax \sin bx \; dx, \quad \int \cos ax \cos bx \; dx$$

Wher $a, b \in \mathbb{Z}$

1. The integral $\int \sin ax \cos bx \; dx$ can be solved using the formula $\sin ax \cos bx = \frac{1}{2}[\sin(ax + bx) + \sin(ax - bx)]$
2. The integral $\int \sin ax \sin bx \; dx$ can be solved using the formula $\sin ax \sin bx = \frac{1}{2}[\cos(ax - bx) - \cos(ax + bx)]$
3. The integral $\int \cos ax \cos bx \; dx$ can be solved using the formula $\cos ax \cos bx = \frac{1}{2}[\cos(ax + bx) + \cos(ax - bx)]$

Integrals Involving Trigonometric Functions (Examples)

1. $\int \sin 3x \cos 2x \ dx = \frac{1}{2} \int [\sin 5x + \sin x] dx =$
$\frac{1}{2} \int \sin 5x \ dx + \frac{1}{2} \int \sin x \ dx = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c$
2. $\int \sin x \sin 3x \ dx = \frac{1}{2} \int [\cos 2x - \cos 4x] dx =$
$\frac{1}{2} \int \cos 2x \ dx - \frac{1}{2} \int \cos 4x \ dx = \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + c$
3. $\int \cos 5x \cos 2x \ dx = \frac{1}{2} \int [\cos 7x + \cos 3x] dx =$
$\frac{1}{2} \int \cos 7x \ dx + \int \cos 3x \ dx = \frac{1}{4} \sin 7x + \frac{1}{6} \sin 3x + c$

Integrals Involving Trigonometric Functions

Second : Integrals of the forms

$$\int \sin^n x \cos^m x \ dx, \quad \int \sinh^n x \cosh^m x \ dx, \qquad Where\ n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd.

1. If n is odd: The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$
2. If m is odd: The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$

Integrals Involving Trigonometric Functions (Examples)

1. $\int \sin^5 x \cos^4 x dx$ to solve this integral put
   $u = \cos x \Rightarrow -du = \sin x \ dx$
   $\int \sin^5 x \cos^4 x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx =$
   $\int (1 - \cos^2 x) \cos^4 x \sin x \ dx = -\int (1 - u^2)^2 u^4 du = -\int u^4 - 2u^6 + u^8 du = -[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}] + c = -[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}] + c$
2. $\int \sin^7 \cos^3 x \ dx$ to solve this integral put
   $u = \sin x \Rightarrow du = \cos x \ dx$
   $\int \sin^7 \cos^3 x \ dx = \int \sin^7 x(1 - \sin^2 x) \cos x \ dx =$
   $\int u^7 (1 - u^2) \ du = \int u^7 - u^9 \ du = \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{sin^{10}}{10} + c$

Integrals Involving Trigonometric Functions (Examples)

- $\int \sinh^3 x \cosh^2 x \; dx$ to solve this integral put
  $u = \cosh x \Rightarrow du = \sinh x$
  $\int \sinh^3 x \cosh^2 x \; dx = \int (\cosh^2 x - 1) \cosh^2 x \sinh x \; dx =$
  $\int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c =$
  $\frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$

Special cases :

1. $\int \sin^2 x \; dx = \frac{1}{2} \int [1 - \cos 2x] \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + c$
2. $\int \cos^2 x \; dx = \frac{1}{2} \int [1 + \cos 2x] \; dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$

Integrals Involving Trigonometric Functions

Third : Integrals of the forms

$$\int \sec^n x \tan^m x \ dx, \quad \int csc^n x \cot^m x \ dx,$$

$$\int sech^n x \tanh^m x \ dx, \quad \int csch^n x \coth^m x \ dx$$

The above four integrals can be solved by substitution if n is even or m is odd.

Integrals Involving Trigonometric Functions

1. If n is even:
   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitution $u = \cot x$, $u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.
2. If m is odd:
   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitutions $u = csc\ x, u = sech\ x$ and $u = csch\ x$ can be used to solve the other theree integrals respictively.

Integrals Involving Trigonometric Functions (Examples)

1. $\int \tan^3 x \sec^3 x \ dx$ to solve this integral put
   $u = \sec x \Rightarrow du = \sec x \tan x \ dx$
   $\int \tan^3 x \sec^3 x \ dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \ dx =$
   $(u^2 - 1)u^2 \ du = \int u^4 - u^2 \ du = \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5}{5} - \frac{\sec^3}{3} + c$
2. $\int tanh^3 x \ sech \ x \ dx$ to solve this integral put
   $u = sech \ x \Rightarrow -du = sech \ x \tanh x \ dx$
   $\int tanh^3 x \ sech \ x \ dx = \int (1 - sech^2 x) sech \ x \ tanh \ x \ dx =$
   $-\int (1 - u^2) du = -u + \frac{u^3}{3} + c = -sech \ x + \frac{sech^3 x}{3} + c$

Trigonometric Substitutions

If the integrand contains a term of the form
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$,
then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$ use the substitution $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$ where $0 \leq \theta < \frac{\pi}{2}$ to solve the integral.

Trigonometric Substitutions (Examples)

Example 4.1

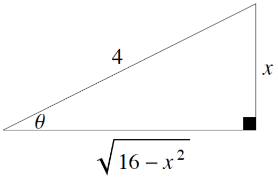
Solve the following integral: $\int \frac{1}{x^2\sqrt{16-x^2}}dx$

$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{x^2\sqrt{(4)^2-x^2}}dx,$ *Put* $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin\theta}}d\theta = \int \frac{4\cos\theta}{16\sin^2\theta \; 4\cos\theta}d\theta$

$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta = \frac{1}{16}\cot\theta + c$

$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c$



Trigonometric Substitutions (Examples)

Example 4.2

Solve the following integral: $\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx$

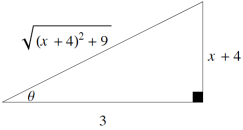
$\int \frac{1}{[(x^2+8x+16)+9]^{\frac{3}{2}}}\ dx = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\ dx.$

Put $x+4 = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\ d\theta$

$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta = \int \frac{3\sec^{\theta}}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta$

$= \int \frac{3\sec^2\theta}{27\sec^3\theta} d\theta = \frac{1}{9}\int \frac{1}{\sec\theta} d\theta = \frac{1}{9}\sin\theta + c$

$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \frac{1}{9}\frac{x+4}{\sqrt{x^2+8x+25}} + c$



Trigonometric Substitutions (Examples)

Example 4.3

Solve the following integral: $\int \frac{\sqrt{x^2-4}}{x^2}\ dx$

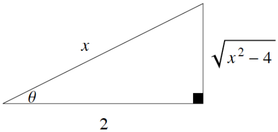
Put $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^2}\ dx = \int \frac{\sqrt{4\sec^2\theta - 4}\ \ 2\sec\theta\tan\theta}{4\sec^2\theta} d\theta = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta} d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta = \int \frac{\sec^2\theta}{\sec\theta} d\theta - \int \frac{1}{\sec\theta} d\theta$$

$$= \int \sec\theta d\theta - \int \cos\theta d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + c$$

$$\int \frac{\sqrt{x^2-4}}{x^2}\ dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$$



Method of Partial fractions

Definition 5.1

Linear Factor:

A linear factor is a polynomial of degree 1. It has the form $ax + b$ *where* $a, b \in \mathbb{R}$ *and* $a \neq 0$.
Such $x, 3x$, *and* $2x - 7$

Definition 5.2

Irreducible Quadratic:

An irreducible quadratic is a polynomial of degree 2. It has the form $ax^2 + bx + c$, *where* $a, b, c \in \mathbb{R}$ *and* $b^2 - 4ac < 0$.
Such $x^2 + 9$ *and* $x^2 + x + 1$.

Method of Partial fractions

What is the Partial Fraction?

It is re-expressing a rational function (a ratio of polynomial function) as a sum of simpler fraction.

Let $h(x) = \frac{P(x)}{Q(x)} = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a - n - 1x + a_n}{b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m}$ be a rational function, where $P(x), Q(x)$ are polynomials, we have two cases:

1. degree $P(x) <$ degree $Q(x)$ use method of partial fractions.
2. degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials, then use method of partial fractions.

Method of Partial fractions

How do we create partial functions?

1. If we can write $Q(x)$ as a linear factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m = (x-a)^m, \; a \in \mathbb{R}, \; m \in \mathbb{N}$
Then: $h(x) = \frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \cdots + \frac{A_{m-1}}{x-a}, m \in \mathbb{N}$
2. If we can write $Q(x)$ as a irreducible quadratic factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m = (ax^2+bx+c)^n, \; a, b, c \in \mathbb{N}$
and $b^2 - 4ac < 0$
Then: $h(x) = \frac{B_0x+C_0}{(ax^2+bx+c)^n} + \frac{B_1x+C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x+C_{n-1}}{ax^2+bx+c}.$

Some time we can write $Q(x)$ as a product of linear factors and irreducible quadratics.

Then $h(x) = \frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \cdots + \frac{A_{m-1}}{x-a}$
$+ \frac{B_0x+C_0}{(ax^2+bx+c)^n} + \frac{B_1x+C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x+C_{n-1}}{ax^2+bx+c}.$

Method of Partial fractions

Example 5.1

Determine the partial fraction for: $\frac{x-3}{x^2-4}$

$\frac{x-3}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow x-3 = A(x+2) + B(x-2)$
$x = -2 \Rightarrow -5 = -4B \Rightarrow B = \frac{5}{4}$
$x = 2 \Rightarrow -1 = 4A \Rightarrow A = \frac{-1}{4}$
So: $\frac{x-3}{(x-2)(x+2)} = \frac{-1}{4(x-2)} + \frac{5}{4(x+2)}$
Now Integrate:
Determine $\int \frac{x-3}{x^2-4} dx$
$\int \frac{x-3}{x^2-4} dx = \int [\frac{-1}{4(x-2)} + \frac{5}{4(x+2)}] dx = -\frac{-1}{4} \ln|x-2| + \frac{5}{4} \ln|x+2| + c$

Method of Partial fractions

Example 5.2

Determine $\int \frac{x-3}{x^2+4x}dx$

Note that degree $P(x) <$ degree $Q(x)$

$\frac{x-3}{x^2+4x} = \frac{x-3}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4} \Rightarrow x - 3 = A(x+4) + Bx$

$x = -4 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$

$x = 0 \Rightarrow -3 = 4A \Rightarrow A = \frac{-3}{4}$

Now we can write $\frac{x-3}{x^2-4x} = \frac{-3}{4x} + \frac{7}{4(x+4)}$. So

$$\int \frac{x-3}{x^2-4x}dx = \int \frac{-3}{4x}dx + \int \frac{7}{4(x+4)}dx = -\frac{3}{4}\ln|x| + \frac{7}{4}\ln|x+4| + C$$

$$= \frac{\ln|x+4|^{\frac{7}{4}}}{\ln|x|^{\frac{3}{4}}} + C$$

Method of Partial fractions

Example 5.3

Determine $\int \frac{x^2-2}{x^3+3x^2+2x}dx$

Note that degree $P(x) <$ *degree* $Q(x)$

$\frac{x^2-2}{x^3+3x^2+2x} = \frac{x^2-2}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$

$\Rightarrow x^2 - 2 = A(x+2)(x+1) + Bx(x+1) + Cx(x+2)$

$x = 0 \Rightarrow -2 = 2A \Rightarrow A = -1$

$x = -2 \Rightarrow 2 = 2B \Rightarrow b = 1$

$x = -1 \Rightarrow -1 = -C \Rightarrow c = 1$

Now we can write $\frac{x^2-2}{x^3+3x^2+2x} = \frac{-1}{x} + \frac{1}{x+2} + \frac{1}{x+1}$. So

$$\int \frac{x^2-2}{x^3+3x^2+2x}dx = \int \frac{-1}{x}dx + \int \frac{1}{x+2}dx + \int \frac{1}{x+1}$$
$$= -\ln|x| + \ln|x+2| + \ln|x+1| + c$$

Method of Partial fractions

Example 5.4

Determine $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}dx$

Note that degree $P(x) <$ *degree* $Q(x)$.

We can write $x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

$$\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{Ax+b}{x^2+4} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow x^3 - 2x^2 + x + 1 = (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 4)$$

$$= (A + C)x^3 + (B + D)x^2 + (A + 4C)x + (B + 4D)$$

$$A = 1, \quad B = -3, \quad C = 0, \quad D = 1$$

$$\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{x-3}{x^2+4} + \frac{1}{x^2+1}$$

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}dx = \int [\frac{x-3}{x^2+4} + \frac{1}{x^2+1}]dx$$

$$= \frac{1}{2}\ln|x^2 + 4| - \frac{3}{2}\tan^{-1}\frac{x}{2} + \tan^{-1}x + c$$

Method of Partial fractions

Example 5.5

Determine $\int \frac{x^2+3}{x^2-x-2}dx$

Note that degree $P(x) \geq$ *degree* $Q(x)$

Here Divide First

$\frac{x^2+3}{x^2-x-2} = 1 + \frac{x+5}{x^2-x-2}$

$\frac{x+5}{x^2-x-2} = \frac{x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow x+5 = A(x+1) + B(x-2)$

$x = 2 \Rightarrow 7 = 3A \Rightarrow A = \frac{7}{3}$ $x = -1 \Rightarrow 4 = -3B \Rightarrow B = \frac{-4}{3}$

$\int \frac{x^2+3}{x^2-x-2}dx = \int [1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}]dx =$

$= x + \frac{7}{3}\ln|x-2| - \frac{4}{3}\ln|x+1| + c$

Example 5.6

Determine $\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx$

Note that degree $P(x) \geq$ *degree* $Q(x)$

$$\frac{x^4+1}{(x+1)(x^2+x+1)} = (x-2) + \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}$$

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx = \int (x-2)\ dx + \int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}dx$$

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow 2x^2+3x+3 = A(x^2+x+1) + (Bx+C)(x+1)$$

$$= Ax^2 + Ax + A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (A+B+C)x + (A+C)$$

$A + B = 2$

$A + B + C = 3$

$A + C = 3$

So: $A = 2$, $B = 0$, and $C = 1$.

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}\,dx = \int (x-2)\ dx + \int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}\,dx$$
$$= \int (x-2)\ dx + \int \frac{2}{x+1}\,dx + \int \frac{1}{x^2+x+1}\,dx$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{2\tan^{-1}(\frac{2x+1}{\sqrt{3}})}{\sqrt{3}} + c$$

So,

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}\,dx = x^2 - 2x + 2\ln|x+1| + \frac{2\tan^{-1}(\frac{2x+1}{\sqrt{3}})}{\sqrt{3}} + c$$

Exercises

1. $\int \frac{6x+7}{(x+2)^2}\,dx$
2. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$
3. $\int \frac{x}{x^2+2x-3}\,dx$
4. $\int \frac{x^2}{(x-1)^2(x+1)}\,dx$
5. $\int \frac{x^3-5x+7}{x^2+x-6}\,dx$
6. $\int \frac{x^3-11x-26}{x^2-2x-8}\,dx$
7. $\int \frac{1}{x(x^2+1)^2}$