King Saud University: Mathematics Department Math-254
First Semester 1440-41 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:—————————— **I.D. No.** ——————

Name of the Teacher:—————————— **Section No.** ——————

Note: Check the total number of pages are Five (5).

(**20 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.20 : Marks: 1.5 for each one ($1.5 \times 20 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Q. No.	11	12	13	14	15	16	17	18	19	20
a,b,c,d										

Quest. No.	Marks
Q. 1 to Q. 20	
Q. 21	
Q. 22	
Total	

Question 1: The number of bisections required to solve the equation $f(x) = 0$ in $[a, a+1]$ accurate to within 10^{-3} is:

(a) 29 **(b)** 21 **(c)** 10 **(d)** 35

Question 2: The relative error for the second approximation of $(18)^{1/4}$ using bisection method on $[2, 2.5]$ is:

(a) 0.03167 **(b)** 0.0652 **(c)** 0.0302 **(d)** 0.0021

Question 3: The value of k which insures rapid convergence of $x_{n+1} = x_n - k(3 - x_n^2)$ to $\alpha = \sqrt{3}$ is:

(a) $\frac{1}{2\sqrt{5}}$ **(b)** $\frac{1}{2\sqrt{3}}$ **(c)** $-\frac{1}{2\sqrt{5}}$ **(d)** $-\frac{1}{2\sqrt{3}}$

Question 4: Which of the following sequences will converge faster to $\sqrt{5}$:

(a) $x_{n+1} = \frac{1}{3}[3x_n + 1 - \frac{x_n^2}{5}]$ **(b)** $x_{n+1} = x_n + 1 - \frac{x_n^2}{5}$ **(c)** $x_{n+1} = \frac{1}{2}[2x_n + 1 - \frac{x_n^2}{5}]$ **(d)** $x_{n+1} = \frac{5}{x_n}$

Question 5: The first approximation using Newton's method of the intersection point of $f(x) = x^3$ and $g(x) = 2x + 1$ with $x_0 = 1.5$ is:

(a) 1.3684 **(b)** 1.6315 **(c)** 1.4212 **(d)** 1.8721

Question 6: The norm of the Jacobi iteration matrix of the following linear system $6x_1 + 2x_2 = 1, \quad x_1 + 7x_2 - 2x_3 = 2, \quad 3x_1 - 2x_2 + 9x_3 = -1$ is:

(a) $\frac{5}{9}$ **(b)** $\frac{3}{2}$ **(c)** $\frac{2}{5}$ **(d)** $\frac{3}{11}$

Question 7: Let $A = \begin{pmatrix} -4 & 6 \\ -2 & 2 \end{pmatrix}$, then the matrix L of the LU factorization using Crout's method is:

(a) $L = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix}$ **(b)** $L = \begin{pmatrix} -4 & 0 \\ -2 & -1 \end{pmatrix}$ **(c)** $L = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$ **(d)** $L = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$

Question 8: Let $A = \begin{pmatrix} \alpha & 0 \\ 1 & 1 \end{pmatrix}$ and $\alpha > 2$. If the condition number $k(A)$ of the matrix A is 6, then α equals to:

(a) 5 **(b)** 3 **(c)** 4 **(d)** 6

Question 9: If the function $f(x) = \frac{1}{x}$ defined on $[2, 4]$, the approximation of $f(3)$ using a linear interpolation polynomial is:

(a) $\frac{11}{8}$ **(b)** $-\frac{5}{8}$ **(c)** 0 **(d)** $\frac{3}{8}$

Question 10: Using Question 9, the value of unknown η in the error formula is:

(a) $\eta = 2.8845$ **(b)** $\eta = 2.7859$ **(c)** $\eta = 2.6145$ **(d)** $\eta = 2.8541$

Question 11: If $f(x) = e^{-x}$, then the value of the divided difference $f[0, 1, 0]$ is:

(a) 0.368 **(b)** 1.0 **(c)** 0.0 **(d)** -0.632

Question 12: Using data: $(1, 1), (2, 0.67), (3, 0.5), (4, 0.4)$, then the linear spline interpolates $f(2.9)$ is:

(a) 0.373 **(b)** 2.121 **(c)** 1.671 **(d)** 0.517

Question 13: When using the two-point forward difference formula with $h = 0.2$ for approximating the value of $f'(1)$, where $f(x) = \ln(x+1)$, we have the computed approximation (accurate to 4 decimal places) as:

(a) 0.4666 **(b)** 0.4966 **(c)** 0.4766 **(d)** 0.4866

Question 14: Using three-point formula for $f''(x)$, the estimate value of $f(0.1)$ with stepsize $h = 0.1$, where $f''(x) = x^2 f(x),\ f(0) = 1,\ f(0.2) = 3$ is:

(a) 1.9999 **(b)** 2.9999 **(c)** 4.9999 **(d)** 3.9999

Question 15: If $f(x) = x^2 + \cos x$ (x in radian), then the approximation of $f''(1)$ with stepsize $h = 0.1$ is:

(a) 1.16 **(b)** 1.46 **(c)** 1.26 **(d)** 1.56

Question 16: The error bound for the approximation in Question 15 is:

(a) 0.00083 **(b)** 0.00037 **(c)** 0.00026 **(d)** 0.00054

Question 17: If $f(0) = 3, f(1) = \dfrac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\displaystyle\int_0^2 f(x)\ dx$ gives 2, then the value of α is:

(a) 0.5 **(b)** 2.0 **(c)** 1.0 **(d)** 3.0

Question 18: The approximation of $\displaystyle\int_0^1 e^{4x}\ dx$ using composite Trapezoidal's rule with $n = 4$ is:

(a) 14.6980 **(b)** 14.4980 **(c)** 14.3980 **(d)** 14.2980

Question 19: Given $xy' + y = 1,\ y(1) = 0$, the approximate value of $y(2)$ using Euler's method when $n = 1$ is:

(a) 1.0 **(b)** 0.0 **(c)** 2.0 **(d)** 3.0

Question 20: Given $y' - \frac{1}{3y} = 0,\ y(0) = 1$, the approximate value of $y(1)$ using Taylor's method of order 2 when $n = 1$ is:

(a) $\frac{25}{18}$ **(b)** $\frac{23}{18}$ **(c)** $\frac{19}{18}$ **(d)** $\frac{17}{18}$

Question 21: *(a)* Find approximation of $\int_1^2 f(x)\,dx$, using best integration rule: (5)

x	1.0	1.1	1.2	1.3	1.4	1.5	1.7	1.8	1.9	2.0
$f(x)$	0.368	0.366	0.361	0.354	0.355	0.335	0.311	0.298	0.284	0.271

(b) The function tabulated is $f(x) = xe^{-x}$, compute an error bound for the approximation using integration rule used in part *(a)*.

(c) How many subintervals approximate the given integral to an accuracy of at least 10^{-6} ?

Solution. *(a)* Using equally spaced points $x_0 = 1.0, x_1 = 1.5, x_2 = 2.0$, gives $h = 0.5$, that is, we select the following table:

x	1.0	1.5	2.0
$f(x)$	0.368	0.335	0.271

So the Simple Simpson's rule for three points can be used and it written as

$$\int_1^2 f(x)\,dx \approx S_2(f) = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big],$$

$$\int_1^2 f(x)\,dx \approx 0.1667[0.368 + 4(0.335) + 0.271] = 0.3299.$$

(b) The derivatives of the function $f(x) = xe^{-x}$ can be obtain as

$$f'(x) = (1-x)e^{-x}, \quad f''(x) = (x-2)e^{-x}, \quad f'''(x) = (3-x)e^{-x}, \quad f^{(4)}(x) = (x-4)e^{-x}.$$

Since $\eta(x)$ is unknown point in $(1,2)$, therefore, the bound $|f^{(4)}|$ on $[1,2]$ is

$$M = \max_{1 \le x \le 2} |f^{(4)}(x)| = \max_{1 \le x \le 2} |(x-2)e^{-x}| = 1.1036,$$

at $x = 1$. Thus the error formula becomes

$$|E_{S_2}(f)| \le \frac{(0.5)^5}{90}(1.1036) = 3.8319e - 004,$$

which is the possible maximum error in our approximation.

(c) To find the minimum subintervals for the given accuracy, we use the error bound formula for Simple Simpson's rule such that

$$|E_{S_2}(f)| \le \frac{|-h^5|}{90} M \le 10^{-6},$$

or

$$n \ge \left(\frac{M \times 10^6}{90}\right)^{1/5},$$

where $h = (b-a)/n$. Since $M = 1.1036$, then solving for n, we obtain,

$$n \ge 6.5722.$$

Hence to get the required accuracy, we need 8 subintervals or 9 points. •

Question 22: Use the Taylor's method of order 4 to find the approximate value of $y(1)$ for the given initial-value problem (5)

$$4y' - y = 0, \quad 0 \leq x \leq 1, \quad y(0) = 1, \quad \text{with} \quad n = 2.$$

Compare your approximate solution with the exact solution $y(x) = e^{x/4}$.

Solution. Since the Taylor's method of order 4 is

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!}f'(x_i, y_i) + \frac{h^3}{3!}f''(x_i, y_i) + \frac{h^4}{4!}f'''(x_i, y_i),$$

for $i = 0, 1, \ldots, n-1$.

Given values $x_0 = 0,\ y_0 = 1$ and $f(x, y) = \frac{y}{4}$, we need derivatives of f as follows

$$f'(x, y) = \frac{y}{16}, \qquad f''(x, y) = \frac{y}{64}, \qquad f'''(x, y) = \frac{y}{256}.$$

So using these values we obtain Taylor's method of order 4 of the form

$$y_{i+1} = y_i \left[1 + \frac{h}{4} + \frac{h^2}{32} + \frac{h^3}{384} + \frac{h^4}{2072}\right].$$

Then for $i = 0, 1$ and by taking $y_0 = 1,\ h = 0.5$, we get

$$y(0.5) \approx y_1 = 1(1 + 0.125 + 0.0078 + 0.0003 + 0.00003) = 1.13313,$$

$$y(1) \approx y_2 = y_1(1 + 0.125 + 0.0078 + 0.0003 + 0.00003) = 1.13313(1.13313) = 1.2840.$$

Thus $|y(1) - y_2| = |1.2840 - 1.2840| = 0.0000$, is the absolute error. •

Solution of the Final Examination

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a,b,c,d	b	d	a	a	c	c	a	b	b	c

Q. No.	11	12	13	14	15	16	17	18	19	20
a,b,c,d	c	b	a	b	c	d	a	a	b	a

The Answer Tables for Q.1 to Q.20 : MAth

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a,b,c,d	a	c	b	d	a	b	c	d	a	b

Q. No.	11	12	13	14	15	16	17	18	19	20
a,b,c,d	b	a	b	d	a	d	b	d	c	a

The Answer Tables for Q.1 to Q.20 : MATh

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Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	c	a	d	b	b	a	b	a	d	a

Q. No.	11	12	13	14	15	16	17	18	19	20
a,b,c,d	a	d	c	a	b	d	c	b	a	b