1. [2 + 2 + 2 + 2]

   (a) Prove that the set of natural number $\mathbb{N}$ is not bounded above.

   (b) Let A be a non-empty subset of $\mathbb{R}$ that is bounded below, and $\alpha = \inf A$. Show that:

   $$\forall \epsilon > 0, \quad \exists a \in A \quad \text{such that} \quad a < \alpha + \epsilon.$$

   (c) Prove that if $\lim_{n\to\infty} x_n = x > 0$, then there exists a natural number N such that

   $$x_n \geq \frac{x}{2}, \quad \forall\, n \geq N.$$

   (d) Let $0 < b < 1$. Prove that:

   $$\lim_{n\to\infty} nb^n \to 0.$$

2. [(2 + 2) + 3]

   (a) Prove that the following limits do not exist in $\mathbb{R}$:

      1. $\lim_{x\to 0} \sin \frac{1}{x}$
      2. $\lim_{x\to 0} \frac{x}{|x|}$

   (b) Prove by using the definition that $\lim_{x\to 1} \frac{2x+1}{x+2} = 1$.

3. [2 + (2 + 2 + 2)]

   (a) Show that if a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   (b) Test the following series for convergence:

      1. $^{\infty}_{k=1} \frac{4^k-1}{3^k}$.
      2. $^{\infty}_{k=1} \frac{\cos k}{3^k}$.
      3. $^{\infty}_{k=1} \frac{100^k}{k!}$.

4. [2.5 + 2.5]

   (a) Prove that, if a function f is increasing on (a, b) and not bounded above, then $\lim_{x\to b^-} f(x) = \infty$.

   (b) Suppose the functions f and g are uniformly continuous on a subset D of the real numbers $\mathbb{R}$. Prove that $f + g$ is uniformly continuous on D.

5. [2 + 2 + 2]

   (a) Let $a,\ b \in \mathbb{R}$. Show that

   $$|\sin b - \sin a| \leq |b - a|.$$

   (b) If the function f satisfies $|f(x)| \leq |x|^4$, for all $x \in [-1, 1]$, prove that f is differentiable at 0 and find $f'(0)$.

   (c) Approximate the number $e^{0.05}$ with 4 decimal places after the decimal point.

6. [(2 + 1) + 3]

   (a) Let $f : [a, b] \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} 2, & x \in \mathbb{Q} \cap [a, b] \\ 0, & x \notin \mathbb{Q} \cap [a, b] \end{cases}$$

      1. Find the upper and the lower integral of f over $[a, b]$.
      2. Is f integrable on $[a, b]$?

   (b) Use Riemann sums, to evaluate $\int_0^1 (2x - 1)\, dx$.