INTEGRAL CALCULUS (MATH 106)

Dr. Maamoun TURKAWI

king saud university

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1 Antiderivative and indefinite integral

Weekly Objectives

Week 1: Antiderivative and indefinite integral

The student is expected to be able to:

1. compute indefinte integrals using the definition.
2. compute indefinite integrals using the basic table and some properties.
3. compute indefinite integrals using change of variable.

Antiderivative

Definition 2.1

A function G is called an antiderivative of the function f on the interval I if $G'(x) = f(x)$ for all $x \in I$.

Example 2.1

What is the antiderivative of the function $f(x) = 2x$?
- The antiderivative is $G(x) = x^2 + c$, where c is a constant.

Antiderivative

Theorem 2.1

Theorem Antiderivative Forms:

Let $F(x)$ and $G(x)$ be antiderivatives of a continuous function $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

Indefinite integral

Definition 2.2

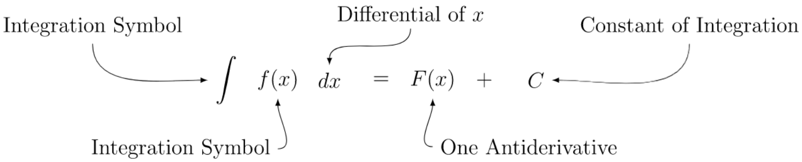
The set of all antiderivatives of $f(x)$ is the ***indefinite integral of f***, *denoted by*

$$\int f(x)\,dx.$$

Given a continuous function f and one of its antiderivatives F, we know all antiderivatives of f have the form $F(x)+C$ for some constant C. Using definition 2.2 , we can say that:

$$\int f(x)\,dx = F(x) + C.$$

Indefinite integral



Indefinite integral: Basic Rules of integration

1. $\int 1dx = x + C$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1, n \in \mathbb{Q}$
3. $\int \cos x \; dx = \sin x + C$
4. $\int \sin x \; dx = -\cos x + C$
5. $\int \sec^2 x \; dx = \tan x + C$
6. $\int \csc^2 x \; dx = -\cot x + C$
7. $\int \sec x \tan x \; dx = \sec x + C$
8. $\int \csc x \cot x \; dx = -\csc x + C$

Indefinite integral: Exemples

1. $$\int 3x^3\,dx = \frac{3}{4}x^4 + C.$$
2. $$\int \sin(\theta)\ d\theta = -\cos(\theta) + C$$
3. $$\int \sec^2(\theta)\ d\theta = \tan(\theta) + C.$$
4. $$\int (\sec(x)\tan(x) + \csc(x)\cot(x))\ dx = \sec(x) - \csc(x) + C$$

Change Of Variable

Example 2.2

Solve $\int (4x+1)^2 \ dx$

Put $u = 4x + 1$ *then* $du = 4dx$ *hence* $\frac{1}{4}du = dx$

$$\int (4x+1)^2 \ dx = \int u^2 \frac{1}{4} du = \frac{1}{4} \int u^2 du = \frac{1}{4} \frac{u^3}{3} + C$$

$$= \frac{1}{4} \frac{(4x+1)^3}{3} + C$$

Change Of Variable

Substitution Rule

$$\int f(g(x))\, g'(x)\; dx = \int f(u)\; du, \qquad \text{where, } u = g(x)$$

Theorem 2.2

Theorem Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\; dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\; dx = \int F'(u)\; du = F(u) + C = F(g(x)) + C.$$

Change Of Variable: Basic Rules of integration

1. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, (n \in \mathbb{Q}, n \neq -1)$
2. $\int \sin(f(x))f'(x)dx = -\cos(f(x)) + C$
3. $\int \cos(f(x))f'(x)dx = \sin(f(x)) + C$
4. $\int \sec^2(f(x))f'(x)dx = \tan(f(x)) + C$
5. $\int \csc^2(f(x))f'(x)dx = -\cot(f(x)) + C$
6. $\int \sec(f(x))\tan(f(x))f'(x)dx = \sec(f(x)) + C$
7. $\int csc(f(x))\cot(f(x))f'(x)dx = -\csc(f(x)) + C$

Change Of Variable: Exemples

Exemple:

$$\begin{aligned}\int (20x+30)(x^2+3x-5)^9\ dx &= \int 10(2x+3)(x^2+3x-5)^9\ dx \\ &= \int 10(\underbrace{x^2+3x-5}_{u})^9 \underbrace{(2x+3)\ dx}_{du} \\ &= \int 10u^9\ du \\ &= u^{10}+C \quad (\text{replace } u \text{ with } x^2+3x-5) \\ &= (x^2+3x-5)^{10}+C\end{aligned}$$

Change Of Variable: Exemples

Exemple: Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

$$u = 1 - 4x^2 \;\Rightarrow\; du = -8x\,dx \;\Rightarrow\; xdx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= \int x\left(1-4x^2\right)^{-\frac{1}{2}}\,dx \\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{4}u^{\frac{1}{2}} + c \\ &= -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} + c\end{aligned}$$

Change Of Variable: Exemples

Exemple: Evaluate $\int \sin(1-x)\,(2-\cos(1-x))^4\,dx$

$$u = 2-\cos(1-x) \;\Rightarrow\; du = -\sin(1-x)\,dx \;\Rightarrow\; \sin(1-x)\,dx = -du$$

The integral is then,

$$\begin{aligned}\int \sin(1-x)\,(2-\cos(1-x))^4\,dx &= -\int u^4\,du\\ &= -\frac{1}{5}u^5 + c\\ &= -\frac{1}{5}(2-\cos(1-x))^5 + c\end{aligned}$$

Change Of Variable: Exemples

Exemple: Evaluate $\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx$

$$u=\cos^2(2x)-5 \;\Rightarrow\; du=-4\cos(2x)\sin(2x)\,dx$$

Then: $10\cos(2x)\sin(2x)\,dx=-\frac{5}{2}du$

Doing the substitution and evaluating the integral gives,

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx=-\frac{5}{2}\int u^{\frac{1}{2}}\,du$$

$$=-\frac{5}{3}u^{\frac{3}{2}}+c$$

Finally, don't forget to go back to the original variable!

$$\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)-5}\,dx=-\frac{5}{3}\left(\cos^2(2x)-5\right)^{\frac{3}{2}}+c$$

Properties of indefinite integral

1. $\int af(x)dx = a\int f(x)dx$ where $a \in \mathbb{R}$
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Remark 2.1

1. $\int f(x)dx = G(x) + C, \int \frac{d}{dx}G(x)dx = G(x) + C$
2. $\frac{d}{dx}\int f(x)dx = f(x)$