# INTEGRAL CALCULUS (MATH 106)

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1. Area Between Curves

2. Volume Of A Solid Revolution

# Weekly Objectives

Week 10: Area between curves and Volume of a solid revolution.

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The student is expected to be able to:

1. Calculate the area between curves.
2. Calculate the volume of a solid revolution using the disk method.
3. Calculate the volume of a solid revolution using the washer method.

# Area Between Two Curves

In this section we are going to look at finding the area between two curves.

## QUISTION:

How we can determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$

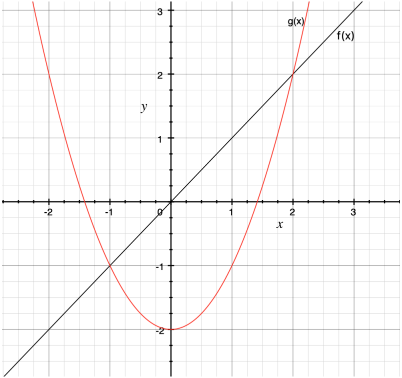
## Theorem: Area Between Curves

Let $f(x)$ and $g(x)$ be continuous functions defind on $[a, b]$ where $f(x) \geq g(x)$ for all $x$ in $[a, b]$.

**The area of the region** bounded by the curves $y = f(x), \quad y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \Big[f(x) - g(x)\Big] \, dx$$

$$A = \int_a^b \text{(upper function) - (lower function) } dx, \qquad a \le x \le b$$




$$A = \int_a^b f(x) - g(x) \; dx$$

# The steps to calculate the area between curves

1. Find the intersection points between the curves.
2. determinant the upper function and the lower function.
3. Calculate the integral:

$$A = \int_a^b \text{(upper function) - (lower function)}\ dx$$

Which give us the required area.

# Area Between Two Curves (Example)

## Example 2.1

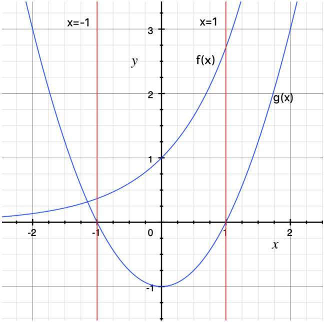
*Find the area enclosed between the graphs $y = x$ and $y = x^2 - 2$.*

1. Points of intersection between $y = x^2 - 2$ and $y = x$ is:
   $x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0$
   $\Rightarrow x = -1$ and $x = 2$
2. Note that upper function is $y = x$ and lower function is $y = x^2 - 2$ Note that $y = x^2 - 2$ is a parabola opens upward with vertex $(0, -2)$, and $y = x$ is a straight line passing through the origin.
3. $A = \int_{-1}^{2} x - (x^2 - 2)\ dx = \int_{-1}^{2} x - x^2 + 2\ dx =$
   $\left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} = \frac{27}{6}$

# Area Between Curves (Example)

## Example 2.2

*Find the area enclosed between the graphs*
$y = e^x, y = x^2 - 1, x = -1,$ *and* $x = 1$



# Area Between Curves (Example)

Note that upper function is $y = e^x$ and lower function is $y = x^2 - 1$

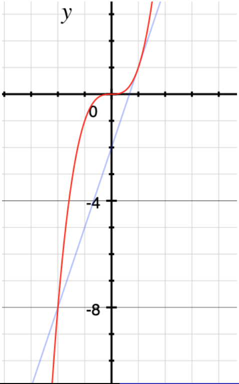
$$A = \int_{-1}^{1} e^x - (x^2 - 1)\ dx = \int_{-1}^{1} e^x - x^2 + 1\ dx = \left[e^x - \frac{1}{3}x^3 + x\right]_{-1}^{1}$$

$$= e - \frac{1}{e} + \frac{4}{3}$$

# Area Between Curves (Example)

## Example 2.3

*Compute the area oh the region bounded by the curves* $y = x^3$ *and* $y = 3x - 2$



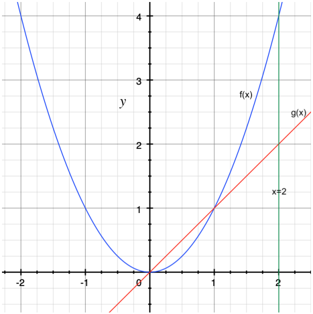
# Area Between Curves (Example)

1. Points of intersection between $y = x^3$ and $y = 3x - 2$
   $x^3 - 3x + 2 = 0 \Rightarrow (x-1)(x^2 + x - 2) = 0$
   $\Rightarrow x = -2$ and $x = 1$
2. Note that upper function is $y = x^3$ and lower function is $y = 3x - 2$
3. $$A = \int_{-2}^{1} x^3 - (3x - 2)\ dx = \int_{-2}^{1} x^3 - 3x + 2\ dx$$
   $$= \left[ \frac{x^4}{4} - \frac{3}{2}x^2 + 2x \right]_{-2}^{1}$$
   $$= \frac{3}{4} + 6 = \frac{27}{4}$$

# Area Between Curves (Example)

## Example 2.4

*Find the area enclosed between the graphs* $f(x) = x^2$ *and* $g(x) = x$ *between* $x = 0$, *and* $x = 2$.



# Area Between Curves (Example)

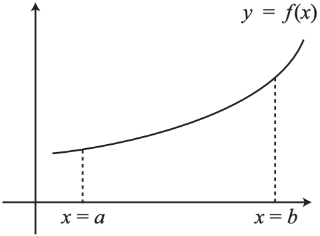
1. we see that the two graphs intersect at $(0,0)$ and $(1,1)$.
2. In the interval $[0,1]$, we have $g(x) = x \geq f(x) = x^2$, and in the interval $[1,2]$, we have $f(x) = x^2 \geq g(x) = x$
3. Therefore the desired area is:

$$A = \int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{0}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{6} + \frac{5}{6} = 1$$

# Volume Of A Solid Revolution (The Disk Method)

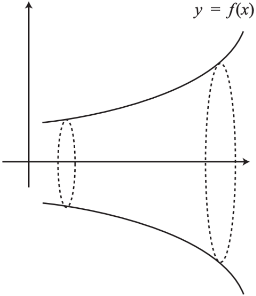
Suppose we have a curve $y = f(x)$




Imagine that the part of the curve between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through 360 degree.

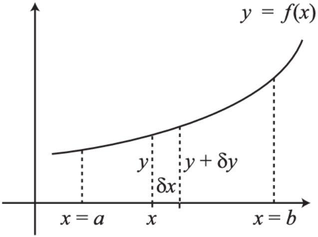
# Volume Of A Solid Revolution (The Disk Method)

Now if we take a cross-section of the solid, parallel to the y-axis, this cross-section will be a circle.




But rather than take a cross-section, let us take a thin disc of thickness $\delta x$, with the face of the disc nearest the y-axis at a distance x from the origin.

# Volume Of A Solid Revolution (The Disk Method)




The radius of this circular face will then be y. The radius of the other circular face will be $y + \delta y$, where $\delta y$ is the change in $y$ caused by the small positive increase in $x, \delta x$.

# Volume Of A Solid Revolution (The Disk Method)

The volume $\delta V$ of the disc is then given by the volume of a cylinder, $\pi r^2 h$, so that

$$\delta V = \pi r^2 \delta x$$

So the volume V of the solid of revolution is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \delta V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \pi \int_a^b [f(x)]^2 \, dx$$

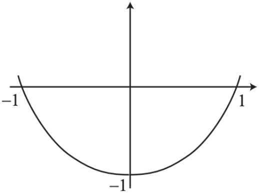
## Example 3.1

*The curve $y = x^2 - 1$ is rotated about the x-axis through 360 degree. Find the volume of the solid generated when the area contained between the curve and the x-axis is rotated about the x-axis by 360 degree.*

$$V = \pi \int_a^b [f(x)]^2 \, dx = \pi \int_{-1}^{1} [x^2 - 1]^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

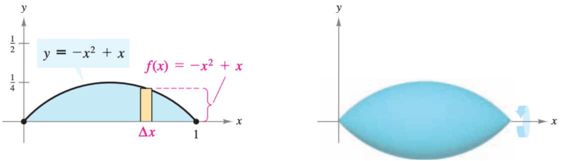
$$= \left[ \frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{-1}^{1} = \frac{16\pi}{15}$$




The graph of $y = x^2 - 1$

## Example 3.2

*Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis.*




Using the Disk Method, you can find the volume of the solid of revolution.

$$V = \pi \int_0^1 [f(x)]^2 \, dx = \pi \int_0^1 [(-x^2 + x)^2 \, dx = \pi \int_0^1 (x^4 - 2x^3 + x^2) \, dx$$

$$= \pi \left[ \frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{30}$$

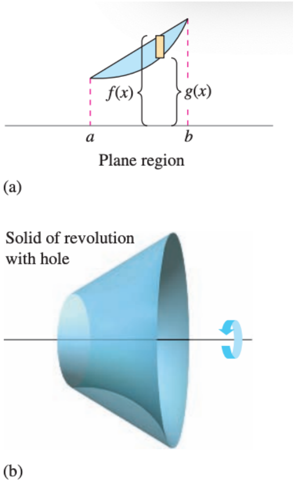
# Volume Of A Solid Revolution (The Washer Method)

## The Washer Method

Let $f$ and $g$ be continuous and nonnegative on the closed interval $[a, b]$, if $f(x) \geq g(x)$ for all $x$ in the interval, then the volume of the solid formed by revolving the region bounded by the graphs of $f(x)$ and $g(x)$ $(a \leq x \leq b)$, about the x-axis is:

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

$f(x)$ is the **outer radius** and $g(x)$ is the **inner radius**.

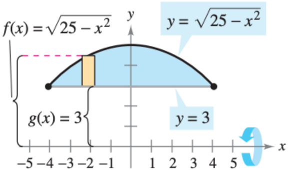


# Volume Of A Solid Revolution (The Washer Method)

## Example 3.3

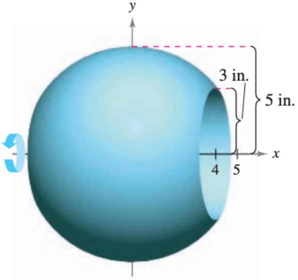
*Find the volume of the solid formed by revolving the region bounded by the graphs of* $f(x) = \sqrt{25 - x^2}$ *and* $g(x) = 3$

We sketch the bounding region and the solid of revolution:



Plane region

(a)



Solid of revolution

(b)

# Volume Of A Solid Revolution (The Washer Method)

First find the points of intersection of $f$ and $g$, by setting $f(x)$ equal to $g(x)$ and solving for $x$.

$\sqrt{25 - x^2} = 3 \Rightarrow 25 - x^2 = 9 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_{-4}^{4} (\sqrt{25 - x^2})^2 - (3)^2 \; dx$$

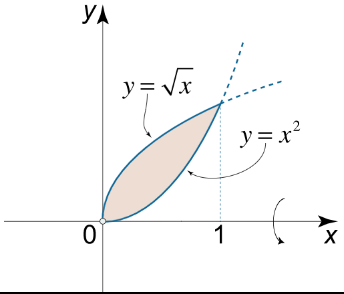
$$= \pi \int_{-4}^{4} (16 - x^2) \; dx = \pi \left[ 16x - \frac{x^3}{3} \right]_{-4}^{4} = \frac{256\pi}{3}$$

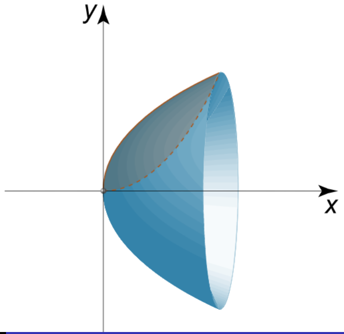
# Volume Of A Solid Revolution (The Washer Method)

## Example 3.4

*Calculate the volume of the solid obtained by rotating the region bounded by the parabola* $y = x^2$ *and the square root function* $y = \sqrt{x}$ *around the* $x-$*axis*

We sketch the bounding region and the solid of revolution:



# Volume Of A Solid Revolution (The Washer Method)

Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 \; dx$$

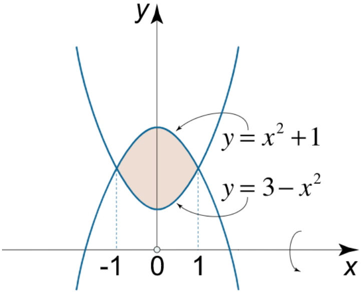
$$= \pi \int_0^1 (x - x^4) \; dx = \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left[ \frac{1}{2} - \frac{1}{5} \right] = \frac{3\pi}{10}$$

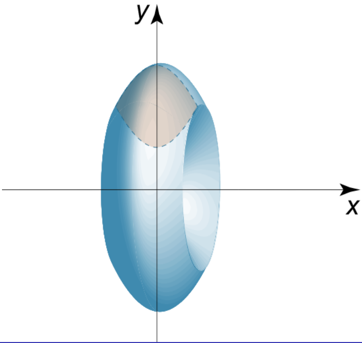
# Volume Of A Solid Revolution (The Washer Method)

## Example 3.5

*Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.*

We sketch the bounding region and the solid of revolution:



# Volume Of A Solid Revolution (The Washer Method)

First we determine the boundaries $a$ and $b$:

$x^2 + 1 = 3 - x^2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Hence the limits of integration are $a = 1$ and $b = -1$.

Using the washer method, we find the volume of the solid:

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

$$= \pi \int_{-1}^{1} \left[ (3 - x^2)^2 - (x^2 + 1)^2 \right] dx = \pi \int_{-1}^{1} \left( 8 - 8x^2 \right) dx$$

$$= 8\pi \int_{-1}^{1} \left( 1 - x^2 \right) dx = 8\pi \left[ x - \frac{x^3}{3} \right]_{-1}^{1} = \frac{32\pi}{3}$$